ANTIPODAL MANIFOLDS IN COMPACT SYMMETRIC SPACES OF RANK ONE

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Abstract. Let $M$ be a compact Riemannian globally symmetric space of rank one. A theorem due to Helgason states that the antipodal manifold $A_x$ of a point $x \in M$ is again a symmetric space of rank one. We compute the multiplicities of the restricted roots of $A_x$ from those of $M$, obtaining a very convenient way to determine $A_x$.

1. Introduction. Let $M$ be a compact Riemannian globally symmetric space of rank one. Let $L$ denote the diameter of $M$ and if $x \in M$ let $A_x$ denote the corresponding antipodal manifold, that is the set of points $y \in M$ at distance $L$ from $x$; $A_x$ is indeed a manifold and with the Riemannian structure induced by $M$, is a symmetric space of rank one totally geodesic in $M$ (see Theorem 2.5). Associated to $M$ there is a triple of numbers $(p, q, \lambda)$ (see below for the definition) which appear in different places, for example in the expressions, in geodesic polar coordinates, of the Riemannian measure and the Laplace-Beltrami operator of $M$. Sometimes doing analysis on these spaces it is necessary to know $p_1$, $q_1$ and $\lambda_1$, the values of $p$, $q$ and $\lambda$ for an antipodal manifold of $M$ (see [2] and [4]). The complete list of compact Riemannian globally symmetric spaces of rank one, with the corresponding values of $p$, $q$ and $\lambda$ is (see [4, p. 171]):

- The spheres $S^n$, $n = 1, 2, \ldots : p = 0, q = n - 1, \lambda = \pi/2L$.
- The real projective spaces $\mathbb{P}^n(\mathbb{R})$, $n = 2, 3, \ldots : p = 0, q = n - 1, \lambda = \pi/4L$.
- The complex projective spaces $\mathbb{P}^n(\mathbb{C})$, $n = 4, 6, \ldots : p = n - 2, q = 1, \lambda = \pi/2L$.
- The quaternion projective spaces $\mathbb{P}^n(\mathbb{H})$, $n = 8, 12, \ldots : p = n - 4, q = 3, \lambda = \pi/2L$.
- The Cayley projective space $\mathbb{P}^{16}(\text{Cay})$: $p = 8, q = 7, \lambda = \pi/2L$.

The superscripts denote the real dimension. The corresponding antipodal manifolds are also known ([1, pp. 437-467], [5, pp. 35 and 52]), but the computations involved are not simple. In this paper we compute $p_1$, $q_1$ and $\lambda_1$ directly from $p$, $q$ and $\lambda$. Since the triple $(p, q, \lambda)$ characterizes $M$, we also obtain a very convenient way to determine the antipodal manifolds of $M$. 
2. We assume \( \dim M > 1 \). This is the case which interests us, and it has the convenient implication that the group \( I(M) \) of isometries of \( M \), in the compact open topology, is semisimple. Let \( o \) be a fixed point in \( M \) and \( s_o \) the geodesic symmetry of \( M \) with respect to \( o \). Let \( U \) denote the identity component of \( I(M) \), \( u \) the Lie algebra of \( U \) and \( u = \mathfrak{f} + \mathfrak{p} \) the decomposition of \( u \) into eigenspaces of the involutive automorphism \( d\gamma \) of \( u \) which corresponds to the automorphism \( \gamma: u \to s_o u s_o \) of \( U \). Here \( \mathfrak{f} \) is the Lie algebra of the subgroup \( K \) of \( U \) which leaves \( o \) fixed. Changing the distance function \( d \) on \( M \) by a constant factor we may assume that the differential of the mapping \( \pi: u \to u \cdot o \) of \( U \) onto \( M \) gives an isometry of \( \mathfrak{p} \) (with the metric of the negative of the Killing form of \( u \)) onto \( M_\mathfrak{p} \), the tangent space to \( M \) at \( o \).

Let \( X \mapsto \text{ad}(X) \) denote the adjoint representation of \( u \). Select a vector \( H \in \mathfrak{p} \) of length \( L \). The space \( \mathfrak{a} = \mathbb{R} H \) is a Cartan subalgebra of the symmetric space \( M \) and we can select a positive restricted root \( \alpha \) of \( M \) such that \( \frac{1}{2} \alpha \) is the only other possible positive restricted root. This means that the eigenvalues of \( (\text{ad} H)^2 \) are 0, \( \alpha(H) \) and possibly \( (\frac{1}{2} \alpha(H))^2 \); \( \alpha(H) \) is purely imaginary. Let \( \mathfrak{u} = \mathfrak{u}_0 + \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha/2} \) be the corresponding decomposition of \( u \) into eigenspaces and \( \mathfrak{f}_\beta = \mathfrak{u}_\beta \cap \mathfrak{f} \), \( \mathfrak{p}_\beta = \mathfrak{u}_\beta \cap \mathfrak{p} \) for \( \beta = 0, \alpha, \frac{1}{2} \alpha \). Then \( \mathfrak{p}_0 = \mathfrak{a} \) and \( \mathfrak{f}_\beta = \text{ad} H (\mathfrak{p}_\beta) \) for \( \beta \neq 0 \). We let \( p = \dim \mathfrak{p}_{\alpha/2} \), \( q = \dim \mathfrak{p}_\alpha \) and \( \lambda = |\alpha(H)|/2L \).

The geodesics in \( M \) are all closed and have length \( 2L \) and the exponential mapping \( \text{Exp} \) at \( o \) is a diffeomorphism of the open ball in \( M \) of center 0 and radius \( L \) onto the complement \( M - A_o \) (see [3, Chapter IX, §5]).

**Lemma 2.1.** (i) \( \alpha(H) = \pm \pi i \) if \( \frac{1}{2} \alpha \) is a restricted root; (ii) \( \alpha(H) = \pm \pi i /2 \) if \( H \) is not conjugate to 0; (iii) \( \alpha(H) = \pm \pi i \) if \( \frac{1}{2} \alpha \) is not a restricted root and \( H \) is conjugate to 0.

**Proof.** Considering \( \text{Exp} \) as a map of \( \mathfrak{p} \) onto \( M \) we have that \( \text{Exp} X = o \) for all \( X \in \mathfrak{p} \) of length \( 2L \). Hence \( d\text{Exp}_{2H} \) vanishes identically on the orthogonal complement of \( \mathfrak{a} \) in \( \mathfrak{p} \). This orthogonal complement is precisely \( \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \). Using the formula for \( d\text{Exp}_{2H} \) [3, Theorem 4.1] it follows that \( \alpha(2H) \in \pi i \mathbb{Z} \), and \( \frac{1}{2} \alpha(2H) \in \pi i \mathbb{Z} \) when \( \frac{1}{2} \alpha \) is a restricted root. Thus if \( \frac{1}{2} \alpha \) is a restricted root \( \alpha(H) = \pi n i \). We also have \( |n| = 1 \) because otherwise \( n^{-1}H \) would be conjugate to 0 and of length less than \( L \), which is impossible. This proves (i).

To prove (ii) we observe that \( \alpha(2n^{-1}H) = \pi i \) for some \( n \in \mathbb{Z} \). As before it follows that \( |n| = 2 \). Now the assumption that \( H \) is not conjugate to 0 implies \( |n| = 1 \). For (iii) we also have \( |n| = 2 \). But now \( |n| = 2 \) because \( H \) is conjugate to 0. Q.E.D.

It will be convenient to choose \( H \) so that the above lemma holds with the minus sign. Let \( o_1 \) denote the point \( \text{Exp}(-H) \).

**Proposition 2.2.** Let \( G \) denote the subgroup of \( U \) leaving the point \( o_1 \in M \) fixed. Then \( \gamma(G) = G \).
Proof. We have $G = \exp(-H)K \exp H$ because $\exp(-H) \cdot o = o_1$. If $g = \exp(-H)k \exp H$, $k \in K$, then $\gamma(g) = \exp Hk \exp(-H)$. Hence to prove the assertion it suffices to show that $\exp H \cdot o = o_1$. We have $\text{Exp}(t + 2H) = \exp tH$ for all $t \in \mathbb{R}$, therefore

$$\exp H \cdot o = \text{Exp} H = \exp(-H) = o_1.$$

Q.E.D.

Corollary 2.3. Let $U^1$ denote the identity component of $G$ and $K^1 = U^1 \cap K$. Then $(U^1, K^1)$ is a Riemannian symmetric pair.

Proof. It is enough to check that $(U^1)_0 \subset K^1 \subset U^1$, where $(U^1)_0$ is the set of fixed points of $\gamma$ in $U^1$ and $(U^1)_0$ is the identity component of $U^1$. If $u \in (U^1)_0$ then $u \in (U^1)_0$, hence $u \in K$ because $(U, K)$ is a symmetric pair [3, Theorem 3.3]. The other inclusion follows similarly. Q.E.D.

Let $u^1$ denote the Lie algebra of $U^1$ and let $u^1 = f^1 + p^1$ be the decomposition of $u^1$ into eigenspaces of the involutive automorphism of $u^1$ which corresponds to the automorphism $\gamma: U^1 \to U^1$.

Proposition 2.4. (i) When $\frac{1}{2} \alpha$ is a restricted root we have

$$f^1 = f_0 + f_\alpha, \quad p^1 = p_{\alpha/2}.$$ (2.1)

(ii) If $H$ is not conjugate to 0

$$f^1 = f_0, \quad p^1 = p_{\alpha}.$$ (2.2)

(iii) If $\frac{1}{2} \alpha$ is not a restricted root and $H$ is conjugate to 0, then

$$f^1 = f_0 + f_\alpha, \quad p^1 = \{0\}.$$ (2.3)

Proof. The Lie algebra of $U^1$ is given by

$$u^1 = \{X \in u: \exp(tX) o_1 = o_1 \text{ for all } t \in \mathbb{R}\}.$$ (2.4)

Therefore $X \in u^1$ if and only if $\exp(tX) \exp(-H) o = \exp(-H) o$ for all $t \in \mathbb{R}$, which is equivalent to

$$\exp(\text{Ad}(\exp H) tX) \in K \quad \text{for all } t \in \mathbb{R},$$

where $\text{Ad}$ denotes the adjoint representation of $U$. Hence

$$u^1 = \{X \in u: \text{Ad}(\exp H) X \in f\}, \quad (2.1)$$

Let $X_\beta$ be a vector in $u_\beta$ for $\beta = 0, \alpha$, $\frac{1}{2} \alpha$. A direct computation which is left to the reader, yields

$$\text{Ad}(\exp H)(X_0 + X_\alpha + X_{\alpha/2}) = X_0 + \cosh(\alpha(H))X_\alpha + \cosh\left(\frac{1}{2} \alpha(H)\right)X_{\alpha/2}$$

$$+ \alpha(H)^{-1} \text{ad } H(\sinh(\alpha(H))X_\alpha + 2\sinh\left(\frac{1}{2} \alpha(H)\right)X_{\alpha/2}).$$

Now the proposition follows from (2.1) and Lemma 2.1 by simple inspection. Q.E.D.

Theorem 2.5. (Cf. [4] Proposition 5.1.) When $\frac{1}{2} \alpha$ is a restricted root or $H$ is not conjugate to 0 the orbit $M_1 = U^1 \cdot o$, with the Riemannian structure induced

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by $M$, is a symmetric space of rank one and a totally geodesic submanifold of $M$.

When $\frac{1}{2} \alpha$ is not a restricted root and $H$ is conjugate to 0 the orbit $M_1 = U^1 \cdot o$ reduces to a point \{o\}. In both cases $M_1 = A_{o_1}$.

**Proof.** Let $y \in M_1$. Writing $y = u \cdot o$, $u \in U^1$, we have $s_y(v \cdot o) = uy(u^{-1}v) \cdot o \in M_1$ for $v \in U^1$ (see Proposition 2.2); hence $s_y(M_1) = M_1$. If $y$ denotes the restriction of $s_y$ to $M_1$, then $y$ is an involutive isometry of $M_1$ with $y$ as isolated fixed point. Thus $M_1$ is globally symmetric and $y$ is the geodesic symmetry with respect to $y$.

To prove that $M_1$ is totally geodesic it suffices to show that each $M$-geodesic which is tangent to $A_1$ at $o$ is a path in $A_1$. Let $t \to \text{Exp} \ tX$, $t \in \mathbb{R}$, be one of such geodesics. Then $X \in \mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{u}^1$. Hence $\text{Exp} \ tX = \exp(tX) \cdot o \in M_1$, $t \in \mathbb{R}$, and our geodesic is a path in $M$. Consequently, the first two assertions follow from the definition of rank and Proposition 2.4.

Clearly, $M_1 \subset A_{o_1}$, both submanifolds are connected, and of the same dimension since $A_o = K \cdot o_1$. Hence $M_1 = A_{o_1}$. This completes the proof of the theorem. Q.E.D.

**Proposition 2.6.** Let $N$ be the kernel of the restriction homomorphism $U^1 \rightarrow I(M_1)$. Then $U^1/N$ is naturally isomorphic to the identity component $I_0(M_1)$ of $I(M_1)$.

**Proof.** If dim $M_1 < 1$ then $M_1$ is either a point or $S^1$ (circle) and dim $I_0(M_1) \leq 1$. The proposition is obviously true in these cases. Therefore we now assume dim $M_1 > 1$.

Let $\mathfrak{z}$ be the center of $u^1$, then $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{f}^1) + (\mathfrak{z} \cap \mathfrak{p}^1)$. But $\mathfrak{z} \cap \mathfrak{p}^1 = \{0\}$ because $M_1$ is a symmetric space of rank one totally geodesic in $M$. Thus $\mathfrak{f}^1 \supset \mathfrak{z}$ and $K^1 \supset Z_0$, the identity component of the center $Z$ of $U^1$. Now $Z_0 \subset Z \subset K^1 \subset N$. Hence $\mathfrak{z} \subset \mathfrak{n}$, the Lie algebra of $N$. Therefore we have a surjective Lie algebra homomorphism $u^1/\mathfrak{z} \rightarrow u^1/\mathfrak{n}$, which implies that $U^1/N$ is semisimple since $u^1/\mathfrak{z}$ is semisimple.

We also have $N \subset K^1$, and we want to consider the pair $(U^1/N, K^1/N)$. Given $u \in N$ and $v \in U^1$ we have $\gamma(u)(v \cdot o) = s_o u(\gamma(v) \cdot o) = s_o \gamma(v) \cdot o = v \cdot o$ (see Proposition 2.2), hence $\gamma(N) \subset N$. Therefore $\gamma$ induces an involutive analytic automorphism, denoted also by $\gamma$, of $U^1/N$. We shall prove that $\gamma$ turns $(U^1/N, K^1/N)$ into a symmetric pair. In fact, $K^1/N \subset U^1/N \subset (U^1/N)_{\gamma}$. On the other hand, if $\exp(tX)N \subset (U^1/N)_{\gamma}$, $X \in \mathfrak{u}^1$, then $\exp(-tX)\gamma(\exp tX) \in N$. Thus $-X + d\gamma(X) \in \mathfrak{n} \subset \mathfrak{f}^1$, which implies $X \in \mathfrak{f}^1$. Therefore $((U^1/N)_{\gamma})_0 \subset K^1/N$.

At this point the proposition follows as a consequence of [3, Theorem 4.1]. Q.E.D.

When $I_0(M_1)$ is identified with $U^1/N$ the isotropy subgroup of $I_0(M_1)$ at $o \in M_1$ becomes $K^1/N$ and the decomposition of the Lie algebra of $I_0(M_1)$ under the corresponding involutive automorphism can be written as $u^1/\mathfrak{n} = \mathfrak{f}^1/\mathfrak{n} + \mathfrak{p}^1$. Therefore the constants $p_1$, $q_1$ and $\lambda_1$ associated to $M_1$ can be
computed from \((\text{ad } H_1)^2: \mathfrak{p}^1 \to \mathfrak{p}^1\), where \(H_1 \in \mathfrak{p}^1\) is a vector of length \(L\) (\(L\) is also the diameter of \(M_1\), see Theorem 2.5). In particular we obtain \(\lambda_1 = \lambda\).

**Proposition 2.7.** If \(H\) is not conjugate to 0, \(p_1 = 0\) and \(q_1 = q - 1\).

**Proof.** Since \(H\) and \(H_1\) are conjugate under \(K\), \((\text{ad } H)^2\) and \((\text{ad } H_1)^2\), as linear transformations of \(\mathfrak{p}\), have the same eigenvalues with the same multiplicities. These eigenvalues are 0 and \(\alpha(H)^2\) (see Lemma 2.1) with multiplicities 1 and \(q\), respectively. Now the proposition follows because 0 is an eigenvalue of \((\text{ad } H_1)^2: \mathfrak{p}^1 \to \mathfrak{p}^1\) and \(\dim \mathfrak{p}^1 = \dim \mathfrak{p}_\alpha = q\) (see Proposition 2.4). Q.E.D.

From now on we shall assume that \(\frac{1}{2} \alpha\) is a restricted root.

Let \(\mathfrak{u}_C\) be the complexification of \(\mathfrak{u}\), \(\theta\) the corresponding extension of \(d\theta\) and \(B\) the Killing form of \(\mathfrak{u}_C\). Let \(\mathfrak{h}\) be any maximal abelian subalgebra of \(\mathfrak{u}\) containing \(\alpha\) and let \(\mathfrak{h}_C\), \(\mathfrak{a}_C\) and \(\mathfrak{p}_C\) denote the subspaces of \(\mathfrak{u}_C\) generated by \(\mathfrak{h}\), \(\mathfrak{a}\) and \(\mathfrak{p}\), respectively. Then \(\mathfrak{h}_C\) is a Cartan subalgebra of \(\mathfrak{u}_C\). Now select compatible orderings in the dual spaces of \(\mathfrak{a}\) and \(\mathfrak{h}_C\), respectively. Let \(\Delta\) denote the set of all nonzero roots (of \(\mathfrak{u}_C\) with respect to \(\mathfrak{h}_C\)) and let \(\Delta^+\) denote the set of all positive roots. Now for each \(\lambda \in \Delta\) the linear function \(\lambda^\theta\) defined by \(\lambda^\theta(X) = \lambda(\theta X), X \in \mathfrak{h}_C\), is again a member of \(\Delta\) and \(\lambda(H)\) is either 0, or \(\pm \frac{1}{2} \alpha(H)\), or \(\pm \alpha(H)\) when \(H \in \alpha\). We may assume that \(\lambda(H)\) is equal to \(\beta(H)\) (\(\beta = 0, \alpha\) or \(\frac{1}{2} \alpha\)) whenever \(\lambda \in \Delta^+, H \in \alpha\) being the vector already chosen.

**Lemma 2.8.** For each \(\lambda \in \Delta^+\) such that \(\lambda(H) = \beta(H), \beta \neq 0\), select a nonzero vector \(X_\lambda\) in the corresponding root subspace. Then the \(X_\lambda - \theta X_\lambda\)'s form a basis of \(\mathfrak{p}_\beta + i\mathfrak{p}_\beta\).

**Proof.** Clearly the vectors \(X_\lambda\)'s, \(\tau X_\lambda\)'s form a basis of \(\mathfrak{u}_\beta + i\mathfrak{u}_\beta\), \(\tau\) being the conjugation of \(\mathfrak{u}_C\) with respect to \(u\). When \(\lambda\) runs over the set of all positive roots such that \(\lambda(H) = \beta(H), \beta \neq 0\), \(\tau X_\lambda\) runs over all root subspaces corresponding to all negative roots \(\mu\) such that \(\mu(H) = -\beta(H)\). But this is precisely what happens with the vectors \(\theta X_\lambda\). Thus the vectors \(X_\lambda\)'s and \(\theta X_\lambda\)'s form also a basis of \(\mathfrak{u}_\beta + i\mathfrak{u}_\beta\). Now the assertion is clear. Q.E.D.

**Lemma 2.9.** For all \(Y \in \mathfrak{p}_\beta + i\mathfrak{p}_\beta, \beta \neq 0\), we have that \((\text{ad } Y)^2\alpha_C \subset \alpha_C\).

**Proof.** Let \(\lambda, \mu \in \Delta^+\) such that \(\lambda(H) = \mu(H)\). By the preceding lemma it suffices to prove that

\[
(\text{ad}(X_\lambda - \theta X_\lambda)\text{ad}(X_\mu - \theta X_\mu) + \text{ad}(X_\mu - \theta X_\mu)\text{ad}(X_\lambda - \theta X_\lambda))H \in \alpha_C.
\]

We have

\[
[X_\mu - \theta X_\mu, H] = -\mu(H)X_\mu + \mu^\theta(H)\theta X_\mu = -\mu(H)(X_\mu + \theta X_\mu),
\]

and
\[ \text{ad}(X_{\lambda} - \theta X_{\lambda}) \text{ad}(X_{\mu} - \theta X_{\mu}) H = -\mu(H)[X_{\lambda} - \theta X_{\lambda}, X_{\mu} + \theta X_{\mu}] \]
\[ = -\mu(H)[[X_{\lambda}, X_{\mu}] + [X_{\lambda}, \theta X_{\mu}] - [\theta X_{\lambda}, X_{\mu}] - \theta[X_{\lambda}, X_{\mu}]]. \]

Interchanging $\lambda$ and $\mu$ and adding we obtain
\[ (\text{ad}(X_{\lambda} - \theta X_{\lambda}) \text{ad}(X_{\mu} - \theta X_{\mu}) + \text{ad}(X_{\mu} - \theta X_{\mu}) \text{ad}(X_{\lambda} - \theta X_{\lambda})) H \]
\[ = -2\lambda(H)([[X_{\lambda}, \theta X_{\mu}] - [\theta X_{\lambda}, X_{\mu}]]. \]

Now
\[ [H, [[X_{\lambda}, \theta X_{\mu}]] = (\lambda(H) + \mu(h)(H))[X_{\lambda}, \theta X_{\mu}] = 0 \]
and also $[H, [\theta X_{\lambda}, X_{\mu}]] = 0$. But $[X_{\lambda}, \theta X_{\mu}] - [\theta X_{\lambda}, X_{\mu}] \in \mathfrak{a}_c$, hence it belongs to $\mathfrak{a}_c$, in view of the maximality of $\mathfrak{a}_c$. This proves our assertion and hence the lemma. Q.E.D.

Note that Lemma 2.9 holds even when $\text{rank}(\theta) > 1$.

**Proposition 2.10.** Let $q(Y)$ be the complex number defined by $(\text{ad} Y)^2 H = q(Y) H$, $Y \in \mathfrak{p}_{a/2} + i \mathfrak{p}_{a/2}$. Then $q(Y) = (\pi/2L)^2 B(Y, Y)$ and $(\text{ad} Y)^2 Z = q(Y) Z$ for all $Z \in \mathfrak{a} + \mathfrak{p}_a$.

**Proof.** It is enough to consider $Y \in \mathfrak{p}_{a/2}$. In fact, $q$ is a quadratic form on $\mathfrak{p}_{a/2} + i \mathfrak{p}_{a/2}$, since
\[ (\text{ad}(Y_1 + Y_2)^2 - \text{ad}(Y_1)^2 - \text{ad}(Y_2)^2) H = (\text{ad} Y_1 \text{ad} Y_2 + \text{ad} Y_2 \text{ad} Y_1) H. \]

The identity component $K_1^1$ of $K^1$ acts transitively on any sphere in $\mathfrak{a} + \mathfrak{p}_a$ with center 0. In fact, $\mathfrak{a} + \mathfrak{p}_a$ is orthogonal to $\mathfrak{p}_{a/2}$ and therefore also stable under $K^1$. Moreover, the tangent space to the orbit $\text{Ad}(K_0^1)H$ at the point $H$ is $[t^1, \mathfrak{a}]$ which equals $\mathfrak{p}_a$ (see Proposition 2.4(i)). It follows that $\text{Ad}(K_0^1)H$ is the sphere in $\mathfrak{a} + \mathfrak{p}_a$ of radius $L$ and center 0.

Now take $x \in K_0^1$, $Y \in \mathfrak{p}_{a/2}$, then
\[ \text{ad}(\text{Ad}(x) Y)^2 (\text{Ad}(x) H) = \text{Ad}(x)(\text{ad} Y)^2 H = q(Y) \text{Ad}(x) H. \]
This shows that $(\text{ad} Y)^2$ acts as a scalar transformation on $\mathfrak{a} + \mathfrak{p}_a$, multiplying by $q(Y)$.

If $Y$ has length $L$ the eigenvalues of $(\text{ad} Y)^2$ as a linear transformation of $\mathfrak{p}$ are the same as those of $(\text{ad} H)^2$. Thus $q(Y) = (\pi/2L)^2$ since $\dim(\mathfrak{a} + \mathfrak{p}_a) = q + 1$. From Lemma 2.1 we get $(1/2 \alpha(H))^2 = -(\pi/2)^2$, from where the proposition follows by homogeneity. Q.E.D.

**Theorem 2.11.** If $\frac{1}{2} \alpha$ is a restricted root then $p_0 = p - q - 1$ and $q_0 = q$.

**Proof.** Given $H_1 \in \mathfrak{p}^1 = \mathfrak{p}_{a/2}$ of length $L$ the eigenvalues of $(\text{ad} H_1)^2$ in $\mathfrak{p}$ are $0, -(\pi/2)^2$ and $-\pi^2$ with multiplicities $1, p$ and $q$, respectively. Proposition 2.10 says that $(\text{ad} H_1)^2$ on $\mathfrak{a} + \mathfrak{p}_a$ has $-(\pi/2)^2$ as an eigenvalue of multiplicity $1 + q$. Therefore the eigenvalues of $(\text{ad} H_1)^2$ in $\mathfrak{p}_{a/2}$ are $0, -(\pi/2)^2$ and $-\pi^2$ with multiplicities $1, p - q - 1$ and $q$, respectively. Q.E.D.
Corollary 2.12. The spheres $S^n$ ($n = 1, 2, \ldots$), the real projective spaces $\mathbb{P}^n(\mathbb{R})$, ($n = 2, 3, \ldots$), the complex projective spaces $\mathbb{P}^n(\mathbb{C})$ ($n = 4, 6, \ldots$), the quaternion projective spaces $\mathbb{P}^n(\mathbb{H})$ ($n = 8, 12, \ldots$) and the Cayley projective plane $\mathbb{P}^{16}(\text{Cay})$ are all the Riemannian globally symmetric spaces of rank 1. The superscripts denote the real dimension. The corresponding antipodal manifolds are in the respective cases: A point, $\mathbb{P}^{n-1}(\mathbb{R})$, $\mathbb{P}^{n-3}(\mathbb{C})$, $\mathbb{P}^{n-4}(\mathbb{H})$, $S^n$.

References


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