A LOWER BOUND ON THE NUMBER OF VERTICES OF A GRAPH

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Abstract. In this note, we derive a lower bound for the number of vertices of a graph in terms of its diameter, $d$, connectivity $k$ and minimum degree $\rho$ which is sharper than that of Watkins [1] by an amount $2(\rho - k)$.

Let $G$ be any finite, undirected graph with neither loops nor multiple edges. Let $n$, $\rho$, $k$ and $d$ denote the number of vertices, minimum degree, connectivity and diameter of $G$ respectively. Watkins [1] has proved that if $k > 1$, then $n > k(d - 1) + 2$. He has used Menger’s theorem to obtain the above result. In this note we prove a theorem from which Watkins’ result follows as a corollary. Our proof is simple and elementary. Moreover the lower bound we obtain is sharper than that of Watkins by the amount $2(\rho - k)$.

Theorem 1. If $k > 1$, then

$$n > \begin{cases} 
  k(d - 3) + 2\rho + 2, & \text{if } d > 3, \\
  \rho + 2, & \text{if } d = 2, \\
  2, & \text{if } d = 1.
\end{cases}$$

Proof. Let $a$ and $b$ be two vertices of $G$ at a distance $d$. Let $A_i = \{x \in V(G) | \delta(a, x) = i\}$, $i = 0, 1, \ldots, d$, where $\delta(a, x)$ denotes the length of the shortest path between $a$ and $x$. Clearly, $A_i \cap A_j = \emptyset$ for $i \neq j$, $A_0 = \{a\}$ and $b \in A_d$. Let $d > 3$. If we delete all the vertices in any $A_i$, $1 < i < d - 1$ from $G$, then the remaining graph becomes disconnected. Hence $|A_i| > k$ for $1 < i < d - 1$. Also $|A_0| + |A_1| > \rho + 1$ and $|A_{d-1}| + |A_d| > \rho + 1$. Hence $n > k(d - 3) + 2\rho + 2$. If $d = 2$, then $|A_1| > \rho$ and so $n > \rho + 2$. If $d = 1$, then clearly $n > 2$. Hence the theorem is proved.

Since $\rho > k$, we have Watkins’ result as a direct corollary to the above theorem.

Now we give below examples of two classes of graphs to show that the bounds in our theorem are ‘best possible’.

Example 1 (Watkins [1]). Let $H_1, \ldots, H_{d-1}$ represent disjoint copies of $K_k$, $G$ is formed as follows: Join each vertex of $H_i$ to each vertex of $H_{i+1}$ by an edge $(i = 1, \ldots, d - 2)$; then join a new vertex $u$ to each vertex of $H_1$ by an edge and similarly join a vertex $v$ to each vertex of $H_{d-1}$. The resulting
graph clearly satisfies the bounds in the theorem but does not show the sharpness of our bound as we have here $\rho = k$.

**Example 2.** Let $d$ and $m$ be integers at least 2, and let $G$ be the lexicographic product of the $2d$-circuit with the complete graph $K_m$. Then we have $k = 2m$, $\rho = 3m - 1$ and $n = 2md$. Now $G$ has diameter $d$ and substitution yields $k(d - 3) + 2\rho + 2 = n$.

In this example we have $\rho > k$ and hence our lower bound is sharper than that of Watkins by an amount $2(\rho - k)$.

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**References**