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A LOWER BOUND ON THE NUMBER OF VERTICES OF A GRAPH

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ABSTRACT. In this note, we derive a lower bound for the number of vertices of a graph in terms of its diameter, d, connectivity k and minimum degree ρ which is sharper than that of Watkins [1] by an amount $2(\rho - k)$.

Let G be any finite, undirected graph with neither loops nor multiple edges. Let n, ρ, k and d denote the number of vertices, minimum degree, connectivity and diameter of G respectively. Watkins [1] has proved that if $k \geqslant 1$, then $n \geqslant k(d-1)+2$. He has used Menger's theorem to obtain the above result. In this note we prove a theorem from which Watkins' result follows as a corollary. Our proof is simple and elementary. Moreover the lower bound we obtain is sharper than that of Watkins by the amount $2(\rho - k)$.

THEOREM 1. *If* $k \geqslant 1$, *then*

$$n \geqslant \begin{cases} k(d-3)+2\rho+2, & \text{if } d \geqslant 3, \\ \rho+2, & \text{if } d = 2, \\ 2, & \text{if } d = 1. \end{cases}$$

PROOF. Let a and b be two vertices of G at a distance d. Let $A_i = \{x \in V(G) | \delta(a, x) = i\}$, $i = 0, 1, \ldots, d$, where $\delta(a, x)$ denotes the length of the shortest path between a and x. Clearly, $A_i \cap A_j = \varnothing$ for $i \neq j$, $A_0 = \{a\}$ and $b \in A_d$. Let $d \geqslant 3$. If we delete all the vertices in any A_i, $1 \leqslant i \leqslant d-1$ from G, then the remaining graph becomes disconnected. Hence $|A_i| \geqslant k$ for $1 \leqslant i \leqslant d-1$. Also $|A_0| + |A_1| \geqslant \rho + 1$ and $|A_{d-1}| + |A_d| \geqslant \rho + 1$. Hence $n \geqslant k(d-3) + 2\rho + 2$. If $d = 2$, then $|A_1| \geqslant \rho$ and so $n \geqslant \rho + 2$. If $d = 1$, then clearly $n \geqslant 2$. Hence the theorem is proved.

Since $\rho \geqslant k$, we have Watkins' result as a direct corollary to the above theorem.

Now we give below examples of two classes of graphs to show that the bounds in our theorem are 'best possible'.

EXAMPLE 1 (WATKINS [1]). Let $H_1, \ldots, H_{d-1}$ represent disjoint copies of K_k, G is formed as follows: Join each vertex of H_i to each vertex of H_{i+1} by an edge ($i = 1, \ldots, d-2$); then join a new vertex u to each vertex of H_1 by an edge and similarly join a vertex v to each vertex of H_{d-1}. The resulting

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graph clearly satisfies the bounds in the theorem but does not show the sharpness of our bound as we have here $\rho = k$.

EXAMPLE 2. Let d and m be integers at least 2, and let G be the lexicographic product of the $2d$-circuit with the complete graph K_m. Then we have $k = 2m$, $\rho = 3m - 1$ and $n = 2md$. Now G has diameter d and substitution yields $k(d - 3) + 2\rho + 2 = n$.

In this example we have $\rho > k$ and hence our lower bound is sharper than that of Watkins by an amount $2(\rho - k)$.

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REFERENCES

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