SYMMETRIC AND ORDINARY DIFFERENTIATION

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ABSTRACT. In 1927, A. Khintchine proved that a measurable symmetrically differentiable function \( f \) mapping the real line \( \mathbb{R} \) into itself is differentiable in the ordinary sense at each point of \( \mathbb{R} \) except possibly for a set of Lebesgue measure zero. Here it is shown that this exceptional set is also of the first Baire category; even more, it is shown to be a \( \sigma \)-porous set of E. P. Dolženko.

1. Introduction. Let \( f \) be a real valued function defined on the real line \( \mathbb{R} \). The upper symmetric derivative of \( f \) at \( x \in \mathbb{R} \) is

\[
D^+f(x) \equiv \lim_{h \to 0} \sup \frac{f(x + h) - f(x - h)}{2h}.
\]

The lower symmetric derivative \( D^-f(x) \) of \( f \) at \( x \) is the corresponding limit inferior. If \( D^+f(x) = D^-f(x) \), then this common value is called the symmetric derivative of \( f \) at \( x \) and is denoted \( f^s(x) \).

It is readily observed that the existence of the ordinary derivative \( f'(x) \) implies the existence of \( f^s(x) \), with the two values being equal. Concerning the reverse implication, Khintchine [4] proved: a measurable function \( f: \mathbb{R} \to \mathbb{R} \) has a finite derivative \( f'(x) \) at almost every point where \( D^+f(x) < \infty \).

We observe that the exceptional set in this theorem, although of Lebesgue measure zero, need not be of the first Baire category even if the condition \( D^+f(x) < \infty \) is replaced by the stronger condition that \( f^s(x) \) exists and is finite. To see this, consider \( f \) to be the characteristic function of an additive group \( G \) of real numbers that is of measure zero and of the second category. Then \( f^s(x) = 0 \) for each \( x \in G \), but \( f'(x) \) exists at no point of \( G \). We also observe that the exceptional set in Khintchine’s theorem need not be of the first category when strong continuity conditions are placed on the function. Indeed, let \( Z \) be any bounded subset of \( \mathbb{R} \) that is both of measure 0 and of the second category. According to C. Goffman [3], there exists a bounded measurable subset \( S \) of \( \mathbb{R} \) whose metric density does not exist at any point of \( Z \). Let \( \chi_S \) be the characteristic function of \( S \) and set

\[
f(x) = \int_{-\infty}^{x} \chi_S(t) \, dt.
\]

Then \( f \) is a Lipschitz function (with its symmetric derivates bounded in...
absolute value by 1 everywhere), but $f'(x)$ does not exist at any point of $Z$.

With the two examples of the previous paragraph in mind, we conclude that in order to obtain a category analogue of Khintchine's theorem we must simultaneously impose some sort of continuity conditions on $f$ and replace the condition $D^sf(x) < \infty$ by the stronger condition that $f'(x)$ exists. In so doing, our result (Theorem 3 in §6) says that if $f$ belongs to the function class

$$\mathcal{F} \equiv \{ f : \text{the points of continuity of } f \text{ are dense in } R \},$$

then the existence of $f'(x)$ implies the existence of $f'(x)$ at all but a $\sigma$-porous set of points (see §2 for the definition of these sets).

This theorem is a direct consequence of Theorem 2 in §5, which gives a relationship between the symmetric derivates and the Dini derivates of functions $f$ in $\mathcal{F}$. The proof of Theorem 2 rests on the determination of the size of the difference set $\mathcal{J} - \mathcal{J}$, where

$$\mathcal{J} = \{ x : f(x - h) < f(x) < f(x + h) \text{ for sufficiently small } h > 0 \}$$

and

$$\mathcal{J} = \{ x : f(x - h) < f(x) < f(x + h) \text{ for sufficiently small } h > 0 \}.$$

This determination is made in §4, while §§2 and 3 are devoted to the establishment of some preliminary results that will be used there.

2. Porosity lemmas. The notion of porosity is due to E. P. Dolženko [1]. The porosity of the set $S \subset R$ at the point $x \in R$ is defined to be the nonnegative value

$$\limsup_{r \to 0} \frac{l(x, r, S)}{r},$$

where $l(x, r, S)$ denotes the length of the largest open interval contained in the set $(x - r, x + r) \cap (R - S)$. The set $S$ is called porous if it has positive porosity at each of its points, and it is called $\sigma$-porous if it is a countable union of porous sets. It follows readily from the definition that a $\sigma$-porous set is both of the first category and of measure zero. On the other hand, L. Zajiček [5] has constructed a perfect set of measure zero that is not $\sigma$-porous.

We now introduce some terminology that will be used in the statements of the following two lemmas concerning points at which a set has porosity $< 1/2$: Let $I$ be an open interval and let $s$ be a point of $R$. The reflection of $I$ in $s$ is defined to be the open interval $\{ 2s - x : x \in I \}$; it is called a left [resp., right] reflection of $I$ in $s$ if $s < (a + b)/2$ [resp., $s > (a + b)/2$], where $I = (a, b)$. If $S \subset R$, then an open interval $I'$ is said to be a finite left [resp., right] reflection of $I$ in $S$ if there exists a finite collection $\{ I_1, I_2, \ldots, I_k \}$ of open intervals and a corresponding subset $\{ s_1, s_2, \ldots, s_{k-1} \}$ of $S$ such that $I = I_1$, $I' = I_k$, and $I_{j+1}$ is the left [resp., right] reflection of $I_j$ in $s_j$ for $j = 1, 2, \ldots, k - 1$. Analogously we define what is meant by a point $p'$ being a finite left or right reflection of a point $p$ in $S$. 
Lemma 1. If $S \subset \mathbb{R}$ and $l(0, r, S)/r < 1/2$ for $0 < r < \delta$, then each open interval $(a, b)$ contained in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that contains 0.

Proof. Suppose $I = (a, b) \subset (0, \delta)$. If $a = 0$, then the reflection of $I$ in any point of $S \cap (0, b/2)$ contains 0. Suppose $a \neq 0$. It follows from the hypothesis that there exists a point $s$ in $S \cap (a/2, a)$, and clearly the reflection $(a', b')$ of $(a, b)$ in $s$ is such that $b' > 0$. If $0 \not\in [a', b')$, then reflect $(a', b')$ in some point $s'$ in $S \cap (a'/2, a')$ to obtain an interval $(a'', b'')$ with $b'' > 0$. Since each of the reflected intervals has the same length as $I$, a finite repetition of this process will produce a finite left reflection $(a^*, b^*)$ of $I$ in $S$ with $0 \in [a^*, b^*)$ and $b^* > 0$. Either $(a^*, b^*)$ contains 0 or $a^* = 0$ and the reflection of $(a^*, b^*)$ in any point of $S \cap (0, b^*/2)$ contains 0. Thus the lemma is established in the case $I \subset (0, \delta)$, and the case $I \subset (-\delta, 0)$ is handled similarly.

Lemma 2. If $S \subset \mathbb{R}$ and $l(0, r, S)/r < \lambda < 1/2$ for $0 < r < \delta$, then, for every $\varepsilon > 0$, each point $p$ in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that is contained in $(0, \varepsilon)$ [resp., $(-\varepsilon, 0)$].

Proof. Suppose $p \in (0, \delta)$. It follows from the hypothesis that there exists a point $s$ in $S \cap (p/2, p/2 + \lambda p)$. If $p_1$ is the reflection of $p$ in $s$, then

$$0 < p_1 = 2s - p < 2\lambda p.$$ 

Similarly, we can reflect $p_1$ in some point $s_1$ in $S \cap (p_1/2, p_1/2 + \lambda p_1)$ to obtain a point $p_2$ with

$$0 < p_2 < 2\lambda p_1 < (2\lambda)^2 p.$$ 

Continuing this reflection process inductively, we obtain a sequence $p_1, p_2, \ldots$ of points with $0 < p_n < (2\lambda)^np$ for each index $n$. Then, since $0 < \lambda < 1/2$, it follows that $p_n \downarrow 0$. This establishes the lemma for $p \in (0, \delta)$, and the proof for $p \in (-\delta, 0)$ is similar.

3. Porosity lemmas applied to the sets $S_n$. In the next section, we will prove that the set $\mathcal{G} - \mathcal{G}$ is a $\sigma$-porous set for functions $f \in \mathcal{F}$, and we will find it convenient to use the decomposition $\mathcal{G} = \bigcup_{n=1}^{\infty} S_n$ where

$$S_n = \{ x : f(x - h) < f(x + h) \text{ for } 0 < h < 1/n \}.$$ 

In that proof, we make use of certain properties that an arbitrary function $f: R \to R$ possesses at a point where the porosity of $S_n$ is $< 1/2$. These properties are given in the next two lemmas.

Lemma 3. Let $f: R \to R$ be arbitrary, and let $\mathcal{C}(f)$ denote the set of points where $f$ is continuous. If $l(0, r, S_n)/r < 1/2$ for $0 < r < \delta < 1/n$, then

$$\sup_{-\delta < x < 0} f(x) \leq f(0) \leq \lim_{x \to 0^+} f(x) < f(0) < \lim_{x \to 0^-} f(x) \leq \inf_{0 < x < \delta} f(x).$$

(*)
Proof. By hypothesis, there exists a strictly decreasing sequence $s_1, s_2, \ldots$ of points in $S_n \cap (0, \delta)$ that converges to 0. Since $s_k \in S_n$, we have $f(0) < f(s_k)$ which yields the third inequality in $(\ast)$. A similar proof establishes the second inequality in $(\ast)$.

Now choose a point $c \in \mathcal{C}(f) \cap (0, \delta)$ and let $\varepsilon > 0$ be given. By the continuity of $f$ at $c$, there is an open subinterval $I$ of $(0, \delta)$ such that $f(x) < f(c) + \varepsilon$ for each $x \in I$. According to Lemma 1, there is a finite left reflection $I'$ of $I$ in $S_n$ with $0 \in I'$. Since $I'$ is a left reflection of $I$ in $S_n$, we have $f(x) < f(c) + \varepsilon$ for each $x \in I'$. The last inequality in $(\ast)$ now follows readily, and the first inequality in $(\ast)$ is established in a similar manner. This completes the proof of the lemma.

**Lemma 4.** Let $f: \mathbb{R} \to \mathbb{R}$ be arbitrary. If $l(0, r, S_n)/r < \lambda < 1/2$ for $0 < r < \delta < 1/n$, then

$$\lim_{x \to 0^+} f(x) = \inf_{0 < x < \delta} f(x) \text{ and } \lim_{x \to 0^-} f(x) = \sup_{-\delta < x < 0} f(x).$$ (†)

Proof. Set $\alpha = \inf_{0 < x < \delta} f(x)$ and let $\varepsilon > 0$ be given. Choose $x \in (0, \delta)$ for which $f(x) < \alpha + \varepsilon$. By Lemma 2 there exists a sequence $x = x_1, x_2, \ldots$ of points such that $x_j \to 0$ and $x_{j+1}$ is a left reflection of $x_j$ in $S_n$ for each index $j$. Hence $f(x_{j+1}) < f(x_j)$ for each index $j$ and it follows that $\lim_{j \to \infty} f(x_j) < \alpha + \varepsilon$. This proves the first equality in (†); the proof of the second equality is similar, and the lemma is established.

4. The set $\mathcal{G}^r - \mathcal{G}$.

**Theorem 1.** If $f \in \mathcal{G}$, then $\mathcal{G}^r - \mathcal{G}$ is $\sigma$-porous; indeed, it is a countable union of sets, each of which has porosity $\geq 1/2$ at each of its points.

Proof. Let $H_n$ denote the set of all points in $S_n - \mathcal{G}$ at which $S_n$ has porosity $< 1/2$, and choose a point $\hat{x} \in H_n$. Let $\lambda$ and $\delta$ be numbers such that $l(\hat{x}, r, S_n)/r < \lambda < 1/2$ for $0 < r < \delta < 1/n$. Replacing 0 by $\hat{x}$ in Lemma 3, we have

$$\sup_{0 < h < \delta} f(\hat{x} - h) < \inf_{0 < h < \delta} f(\hat{x} + h).$$ (′)

However, if equality holds in (′), it follows from Lemma 3 that $f(x)$ is continuous at $\hat{x}$. So, replacing 0 by $\hat{x}$ in Lemma 4, we have that

$$\sup_{0 < h < \delta} f(\hat{x} - h) = f(\hat{x}) = \inf_{0 < h < \delta} f(\hat{x} + h),$$

that is, $\hat{x} \in \mathcal{G}$. But this contradicts $\hat{x} \in H_n$ and so the strict inequality must hold in (′).

Now for each rational number $\alpha$ and each positive integer $k$ let $A_{\alpha k}$ denote the set of all points $x \in \mathbb{R}$ for which

$$\sup_{0 < h < 1/k} f(x - h) < \alpha < \inf_{0 < h < 1/k} f(x + h).$$

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From the discussion in the previous paragraph, it follows that $H_n$ is a subset of the countable union of all the sets $A_{ak}$; furthermore, since $C(f)$ is dense in $R$, it is easily observed that each of the sets $A_{ak}$ is an isolated set, more specifically, if $x \in A_{ak}$ then $A_{ak} \cap (x - 2/k, x + 2/k) = \{x\}$. Hence $H_n$ is a countable set, and we can write

$$S_n - 9 = [(S_n - 9) - H_n] \cup \left( \bigcup_{x \in H_n} \{x\} \right),$$

where each set in the union on the right has porosity $> 1/2$ at each of its points. This proves the theorem.

5. Dini derivatives and symmetric derivatives. The upper right and the upper left Dini derivatives of the function $f: R \rightarrow R$ are respectively

$$D^+ f(x) = \lim_{h \rightarrow 0^+} \sup \frac{f(x + h) - f(x)}{h},$$

and

$$D^- f(x) = \lim_{h \rightarrow 0^-} \sup \frac{f(x + h) - f(x)}{h}.$$

The lower Dini derivatives $D_+ f(x)$ and $D_- f(x)$ are the corresponding lim inf's.

**Theorem 2.** If $f \in \mathcal{F}$, then for all but a $\sigma$-porous set of points both of the following equalities hold:

(i) $D^f(x) = \min\{D_+ f(x), D_- f(x)\}$,

(ii) $D^f(x) = \max\{D^+ f(x), D^- f(x)\}$.

**Proof.** We first observe that for each point $x$ we have

$$\min\{D_- f(x), D_+ f(x)\} \leq D^f(x) \leq D^f(x) \leq \max\{D^- f(x), D^+ f(x)\}. \quad (*)$$

Now for each rational number $\alpha$ set

$$N(f, \alpha) = \{x: D_n f(x) > \alpha > \min\{D_+ f(x), D_- f(x)\}\}.$$ 

Then, in light of the first inequality in $(*)$, we see that in order to prove the statement of the theorem concerning equality (i), it suffices to show that each set $N(f, \alpha)$ is $\sigma$-porous. Furthermore, it is sufficient to show that $N(f, 0)$ is $\sigma$-porous because $N(f, \alpha) = N(g, 0)$ for the function $g(x) = f(x) - \alpha x$. But this is a direct consequence of Theorem 1 since $N(f, 0) \subset \mathcal{F} - \mathcal{F}$.

Now, replacing $f(x)$ with $-f(x)$, we use this same argument to establish the statement of the theorem concerning equality (ii); hence, the proof is complete.

6. Consequences of Theorem 2. An immediate consequence of Theorem 2 is the following theorem which says that, for functions in class $\mathcal{F}$, the ordinary
derivative exists at most points where the symmetric derivative exists.

**Theorem 3.** If \( f \in S \), then \( f'(x) \) exists at all but a \( \sigma \)-porous set of points where \( f^s(x) \) exists.

Now, according to the theorem of Khintchine given in §1 of this paper, a measurable symmetrically differentiable function must be differentiable in the ordinary sense almost everywhere and is therefore in class \( S \). This observation together with Theorem 3 yields the following result which was announced in the abstract of this article.

**Theorem 4.** If \( f \) is measurable and symmetrically differentiable on \( R \), then \( f'(x) \) exists for all but a \( \sigma \)-porous set of points.

Although it has been known for some time that a continuous symmetrically differentiable function is differentiable in the ordinary sense at each point of \( R \) except possibly for a set of the first category and measure zero, it was only recently shown that this exceptional set need not be countable. That is, based upon a construction of J. Foran [2], we see that this exceptional set can equal certain perfect sets of measure zero. In this connection, Foran [2] posed the question as to whether each perfect set of measure zero could be such an exceptional set. Theorem 4 provides a negative answer to this question; that is, the non-\( \sigma \)-porous perfect set of measure zero constructed by L. Zajíček [5] cannot be such an exceptional set. Nevertheless, the following specialization of Foran's question remains open:

*Is each perfect \( \sigma \)-porous set precisely the set of points where some continuous symmetrically differentiable function fails to be differentiable in the ordinary sense?*

The final consequence of Theorem 2 that we wish to mention concerns points of density of a measurable set. It is well known that the set of points \( x \) where the symmetric metric density of a measurable set \( M \) exists but the ordinary metric density does not constitutes a set of the first category and measure zero. By considering \( f(x) \) to be an integral of the characteristic function of \( M \), we see that Theorem 3 allows us to say more.

**Theorem 5.** If \( M \) is a measurable subset of \( R \), then for all but a \( \sigma \)-porous set of points \( x \in R \) the existence of the symmetric metric density of \( M \) at \( x \) implies the existence of the ordinary metric density of \( M \) at \( x \).

**References**

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