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PERIODIC SOLUTIONS OF PERTURBED CONSERVATIVE SYSTEMS

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ABSTRACT. The existence of 2π-periodic solutions to the system $x'' + \operatorname{grad} G(x) = p(t, x)$, p being 2π-periodic in t, is established under conditions at infinity on the Hessian matrix of G. The condition used is weaker than earlier known conditions of a similar nature.

Introduction and preliminaries. This note concerns the differential equation

$$x'' + \operatorname{grad} G(x) = p(t, x), \tag{I}$$

where $G \in C^2(R^n, R)$ and $p \in C(R \times R^n, R^n)$ is 2π-periodic in t for each fixed $x \in R^n$. We give conditions "at infinity" on the Hessian matrix of $G(x)$ which imply the existence of at least one 2π-periodic solution of (I) when $p(t, x)$ is uniformly bounded.

The existence of 2π-periodic solutions to the equation

$$x'' + \operatorname{grad} G(x) = f(t) \tag{II}$$

under condition (1.1) below has been the object of several interesting papers. Loud [8] initiated these studies with an investigation of a scalar version of (II). Leach in [7] extended these results. In [6] Lazer and Sánchez, using the Cesari alternative method, proved the following theorem.

THEOREM A [LAZER-SÁNCHEZ]. *Let $f \in C(R, R^n)$ be 2π-periodic. If $G \in C^2(R^n, R)$ and there exist an integer $\tilde{n}$ and numbers p and q such that $\tilde{n}^2 < p \leqslant q < (\tilde{n} + 1)^2$ and if*

$$pI \leqslant \left(\partial^2 G(a)\right) / \left(\partial x_i \partial x_j\right) \leqslant qI \tag{1.1}$$

for all $a \in R^n$, then there exists a 2π-periodic solution of (II).

In other papers Lazer [5], Ahmad [1], and Kannan [3], and others, have extended these studies to show both existence and uniqueness for wider classes of systems. Recently Mawhin [9] has given a proof of Theorem A (also showing uniqueness) based on an abstract result of his which is in turn based upon a simple and elegant application of the contraction mapping principle. All of these papers have used condition (1.1) or a more general version of (1.1). Here we weaken (1.1) to hold "at infinity" and at the same time allow

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our perturbation to depend upon x. Our methods are most closely related to those of Mawhin.

In this note R denotes the real numbers, $|x|$ denotes the Euclidean norm of $x \in R^n$, $n \geqslant 1$ and $|A|$ will be used for the norm of a matrix A. Also I denotes either the $n \times n$ identity matrix or the identity map on a Hilbert space. If A and B are two real $n \times n$ matrices by $A \leqslant B$ we mean $B - A$ is nonnegative definite.

2. Statement and proof of the theorem.

THEOREM 1. *Let $G \in C^2(R^n, R)$ and $p \in C(R \times R^n, R^n)$ with $p(t + 2\pi, x) = p(t, x)$ for all (t, x) and $p(t, x)$ uniformly bounded. Suppose there exist an integer $\tilde{n}$ and positive numbers p, q, and r with $\tilde{n}^2 < p \leqslant q < (\tilde{n} + 1)^2$ such that whenever $a \in R^n$ and $|a| \geqslant r$*

$$pI \leqslant \left(\partial^2 G(a)/\partial x_i \partial x_j\right) \leqslant qI. \tag{III}$$

Then there is at least one 2π-periodic solution of (I).

PROOF. Without loss of generality we may assume grad $G(0) = 0$ since otherwise we could subtract grad $G(0)$ from each side of the equation (I).

Let $S = L^2((0, 2\pi), R^n)$ be the Hilbert space of square integrable R^n valued functions with the usual inner product

$$(u, v) = \int_0^{2\pi} \langle u(t), v(t) \rangle \, dt$$

and norm denoted by $\|\cdot\|$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on R^n.

Let

$$D = \left\{u \in S\colon u, u' \in \mathrm{AC}, u'' \in L^2, u(0) = u(2\pi) \text{ and } u'(0) = u'(2\pi)\right\}$$

and define $L\colon D \to S$ by $Lu = u''$. Here AC means absolutely continuous.

It follows from (III), the fact that the Hessian $H(x)$ is symmetric, and the continuity of the Hessian $H(x) = (\partial^2 G(x)/\partial x_i x_j)$ on $|x| \leqslant r$ that grad G is globally Lipschitzian and there is a number $a > 0$ such that $|\text{grad } G(x)| \leqslant a|x|$ for all $x \in R^n$. Therefore, if we define an operator N on S by $N(u)(t) = \text{grad } G(u(t))$ for $u \in S$ and $t \in [0, 2\pi]$ the operator N will map S continuously into itself. We define $M\colon S \to S$ by $Mu(t) = p(t, u(t))$ for $u \in S$ and $0 \leqslant t \leqslant 2\pi$. It is shown in [**4**, p. 22] that the continuity and boundedness of P implies that M is continuous and maps S into itself. Any 2π-periodic solution of (I) is then a solution of the equation

$$Lx + Nx = Mx \tag{2.1}$$

in S, and conversely.

Let $c = (p + q)/2$; then $-c \in \rho(L)$ (the spectrum of L is $\{-n^2\colon n \in Z\}$) and $(L + cI)^{-1}$ exists as a bounded linear operator on S and it follows from known results concerning Green's functions (generalized to the case of uncoupled equations in R^n) that $(L + cI)^{-1}$ is compact [2, p. 192]. Equation (2.1) is equivalent to

$$x = (L + cI)^{-1}[M + cI - N]x. \tag{2.2}$$

We will use the Schauder fixed point theorem [11, p. 25] to show that (2.2) has a solution. The operator $M + cI - N$ is continuous and maps bounded sets in S into bounded sets. The compactness of $(L + cI)^{-1}$ now implies that the operator T defined by $T = (L + cI)^{-1}(M + cI - N)$ is completely continuous. If we can show that T maps some closed ball in S into itself we are done.

We first compute $\|(L + cI)^{-1}\|$. Because L is selfadjoint and the interval $(-(\tilde{n} + 1)^2, -\tilde{n}^2)$ contains no numbers in the spectrum of L we have as in [9]:

$$\begin{aligned}\|(L + cI)^{-1}\| &= \max\left\{(c - \tilde{n}^2)^{-1}, \left((\tilde{n} + 1)^2 - c\right)^{-1}\right\}\\ &= \left[\min\{\ \dots\ \}\right]^{-1}.\end{aligned}$$

Now let $u \in S$. We estimate $\|(M + cI - N)u\|$. Let $H(a) = (\partial^2 G(a)/\partial x_i \partial x_j)$ and m a number with $m \geqslant |p(t, x)|$ for all t, x. We have

$$\begin{aligned}\|(M + cI - N)u\| &\leqslant \|Mu\| + \|(N - cI)u\|\\ &\leqslant \sqrt{2\pi}\ m + \|(N - cI)u\|.\end{aligned} \tag{2.4}$$

Further:

$$\begin{aligned}\|Nu - cu\|^2 &= \int_0^{2\pi} |\operatorname{grad} G(u(t)) - cu(t)|^2\, dt\\ &= \int_0^{2\pi} \left|\left[\int_0^1 (H(\lambda u(t)) - cI)\, d\lambda\right] u(t)\right|^2 dt\\ &\leqslant \int_0^{2\pi} \left[\int_0^1 |H(\lambda u(t)) - cI|\, d\lambda\right]^2 |u(t)|^2\, dt\end{aligned}$$

by Taylor's theorem and since grad $G(0) = 0$.

We make the observation that whenever $|\lambda u(t)| \geqslant r$ we have as in [9] by the symmetry of the matrix $(\partial^2 G(\lambda u(t))/\partial x_i \partial x_j) = H(\lambda u(t))$:

$$\begin{aligned}|H(\lambda u(t)) - cI| &= \sup_{|y|=1} \langle H(\lambda u(t))y - cy, y\rangle\\ &\leqslant \max\{(q - c), (c - p)\} = \beta.\end{aligned} \tag{2.5}$$

Let $z(\lambda u(t)) = |H(\lambda u(t)) - cI|$ and choose $\varepsilon > 0$. Let $E_1 = \{(\lambda, t)\colon |\lambda u(t)| \leqslant r \text{ and } 0 \leqslant \lambda \leqslant \varepsilon\}$, $E_2 = \{(\lambda, t)\colon |\lambda u(t)| \leqslant r \text{ and } \varepsilon \leqslant \lambda \leqslant 1\}$, and $E_3 = \{(\lambda, t)\colon |\lambda u(t)| > r\}$. We have:

$$\begin{aligned}\|Nu - cu\|^2 &\leqslant \int_0^{2\pi} \left[\int_0^1 z(\lambda u(t))\, d\lambda \right]^2 |u(t)|^2\, dt \\ &= \int \left[\int_{E_1} z(\lambda u(t))\, d\lambda \right]^2 |u(t)|^2\, dt \\ &\quad + \int \left[\int_{E_2} z(\lambda u(t))\, d\lambda \right]^2 |u(t)|^2\, dt \\ &\quad + \int \left[\int_{E_3} z(\lambda u(t))\, d\lambda \right]^2 |u(t)|^2\, dt \\ &\leqslant \varepsilon^2 k^2 \|u\|^2 + k^2 \int_{\{t : |u(t)| \leqslant r/\varepsilon\}} |u(t)|^2\, dt + \beta^2 \|u\|^2 \\ &\leqslant \varepsilon^2 k^2 \|u\|^2 + k^2 r^2/\varepsilon^2 + \beta^2 \|u\|^2 \end{aligned} \tag{2.6}$$

where $k = \sup\{|H(x) - cI| : |x| \leqslant r\}$. If $\|u\| \geqslant rk/\varepsilon^2$ we have from (2.6):

$$\|Nu - cu\|^2 \leqslant \varepsilon^2 (k^2 + 1) \|u\|^2 + \beta^2 \|u\|^2,$$

$$\|Nu - cu\| \leqslant \left[\varepsilon^2 (k^2 + 1) + \beta^2 \right]^{1/2} \|u\|. \tag{2.7}$$

Since $\beta = \max\{(q - c), (c - p)\}$ and

$$\|(L + cI)^{-1}\| = \left[\min\{ \dots \} \right]^{-1}$$

and $c = (p + q)/2$ with $\tilde{n}^2 < p \leqslant q < (\tilde{n} + 1)^2$ it is clear that

$$\beta \|(L + cI)^{-1}\| < 1. \tag{2.8}$$

Thus we may choose $\varepsilon = \varepsilon_1$ so small that whenever $\|u\| \geqslant kr/\varepsilon_1^2$ we have by (2.7) and (2.8) that there is a number d, $0 < d < 1$, such that

$$\left\|(L + cI)^{-1}\right\| \cdot \|Nu - cu\| \leqslant d \|u\|. \tag{2.9}$$

We can now show that the operator $T = (L + cI)^{-1}(M + cI - N)$ maps a closed ball of the form $B_n = \{u \in S : \|u\| \leqslant n\}$, n a positive integer, into itself. If not, then we can find a sequence $\{x_n\}$ in S with $x_n \in B_n$ and $\|Tx_n\| > n$. The sequence $\{x_n\}$ must tend to infinity in norm, since otherwise T would be mapping a bounded set onto an unbounded one. Hence $\|x_n\| \to \infty$ and $\|x_n\| > rk/\varepsilon_1$ for all but finitely many n. By (2.4) and (2.9) we have for large n:

$$\|x_n\| \leqslant n < \|Tx_n\| \leqslant d \|x_n\| + \sqrt{2\pi}\; m \|(L + cI)^{-1}\|$$

and

$$\|x_n\| \leqslant (1 - d)^{-1} \sqrt{2\pi}\; m \|(L + cI)^{-1}\|.$$

This contradicts the unboundedness of the $\{x_n\}$. Hence T maps some ball B_k into itself. By the Schauder theorem there exists $x_0 \in B_k$ with $x_0 = Tx_0$. Hence $x_0 \in D(L)$ and

$$Lx_0 = -Nx_0 + Mx_0.$$

Since $x_0(t)$ is continuous on $[0, 2\pi]$ and

$$x_0''(t) = -\operatorname{grad} G(x_0(t)) + p(t, x_0(t))$$

it follows that $x_0''(t)$ is continuous on $[0, 2\pi]$. The function $x_0(t)$ may now be extended periodically to all of R. This extension is clearly a periodic solution of (I) on all of R. This completes the proof of the theorem.

REMARK 1. The uniform boundedness of $p(t, x)$ was not essential, and it is clear that the theorem remains true if $p(t, x)$ is sublinear, i.e., if $\lim|p(t, x)|/|x| = 0$, as $|x| \to \infty$, the convergence being uniform in t.

REMARK 2. This method is not restricted to the periodic problem, but could also be used to handle other problems.

REMARK 3. Reissig **[10]** has extended Mawhin's approach to the equation

$$x'' + Cx' + \operatorname{grad} G(x) = e(t)$$

with C symmetric. His results can be combined with the methods of this paper to give similar results to our Theorem 1 for the equation

$$x'' + Cx' + \operatorname{grad} G(x) = p(t, x).$$

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