PERIODIC SOLUTIONS OF PERTURBED CONSERVATIVE SYSTEMS

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Abstract. The existence of $2\pi$-periodic solutions to the system $x'' + \text{grad } G(x) = p(t, x)$, $p$ being $2\pi$-periodic in $t$, is established under conditions at infinity on the Hessian matrix of $G$. The condition used is weaker than earlier known conditions of a similar nature.

Introduction and preliminaries. This note concerns the differential equation

$$x'' + \text{grad } G(x) = p(t, x), \quad (I)$$

where $G \in C^2(\mathbb{R}^n, \mathbb{R})$ and $p \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is $2\pi$-periodic in $t$ for each fixed $x \in \mathbb{R}^n$. We give conditions “at infinity” on the Hessian matrix of $G(x)$ which imply the existence of at least one $2\pi$-periodic solution of (I) when $p(t, x)$ is uniformly bounded.

The existence of $2\pi$-periodic solutions to the equation

$$x'' + \text{grad } G(x) = f(t) \quad (II)$$

under condition (1.1) below has been the object of several interesting papers. Loud [8] initiated these studies with an investigation of a scalar version of (II). Leach in [7] extended these results. In [6] Lazer and Sánchez, using the Cesari alternative method, proved the following theorem.

Theorem A [Lazer-Sánchez]. Let $f \in C(\mathbb{R}, \mathbb{R}^n)$ be $2\pi$-periodic. If $G \in C^2(\mathbb{R}^n, \mathbb{R})$ and there exist an integer $\tilde{n}$ and numbers $p$ and $q$ such that

$$\tilde{n}^2 < p < q < (\tilde{n} + 1)^2$$

and if

$$pl < \langle \partial^2 G(a) / (\partial x_i \partial x_j) \rangle < ql \quad (1.1)$$

for all $a \in \mathbb{R}^n$, then there exists a $2\pi$-periodic solution of (II).

In other papers Lazer [5], Ahmad [1], and Kannan [3], and others, have extended these studies to show both existence and uniqueness for wider classes of systems. Recently Mawhin [9] has given a proof of Theorem A (also showing uniqueness) based on an abstract result of his which is in turn based upon a simple and elegant application of the contraction mapping principle. All of these papers have used condition (1.1) or a more general version of (1.1). Here we weaken (1.1) to hold “at infinity” and at the same time allow...
our perturbation to depend upon \( x \). Our methods are most closely related to those of Mawhin.

In this note \( R \) denotes the real numbers, \( |x| \) denotes the Euclidean norm of \( x \in \mathbb{R}^n \), \( n \geq 1 \) and \( |A| \) will be used for the norm of a matrix \( A \). Also \( I \) denotes either the \( n \times n \) identity matrix or the identity map on a Hilbert space. If \( A \) and \( B \) are two real \( n \times n \) matrices by \( A \preceq B \) we mean \( B - A \) is nonnegative definite.

2. Statement and proof of the theorem.

**Theorem 1.** Let \( G \in C^2(\mathbb{R}^n, \mathbb{R}) \) and \( p \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) with \( p(t + 2\pi, x) = p(t, x) \) for all \((t, x)\) and \( p(t, x) \) uniformly bounded. Suppose there exist an integer \( \tilde{n} \) and positive numbers \( p, q, \) and \( r \) with \( \tilde{n}^2 < p < q < (\tilde{n} + 1)^2 \) such that whenever \( a \in \mathbb{R}^n \) and \( |a| > r \)

\[
\rho I < \left( \frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) < qI. \tag{III}
\]

Then there is at least one \( 2\pi \)-periodic solution of (I).

**Proof.** Without loss of generality we may assume \( \nabla G(0) = 0 \) since otherwise we could subtract \( \nabla G(0) \) from each side of the equation (I).

Let \( S = L^2((0, 2\pi), \mathbb{R}^n) \) be the Hilbert space of square integrable \( \mathbb{R}^n \) valued functions with the usual inner product

\[
\langle u, v \rangle = \int_0^{2\pi} \langle u(t), v(t) \rangle \, dt
\]

and norm denoted by \( \| \cdot \| \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual inner product on \( \mathbb{R}^n \).

Let

\[
D = \{ u \in S : u, u' \in AC, u'' \in L^2, u(0) = u(2\pi) \text{ and } u'(0) = u'(2\pi) \}
\]

and define \( L : D \to S \) by \( Lu = u'' \). Here \( AC \) means absolutely continuous.

It follows from (III), the fact that the Hessian \( H(x) \) is symmetric, and the continuity of the Hessian \( H(x) = (\partial^2 G(x)/\partial x_i x_j) \) on \( |x| < r \) that \( \nabla G \) is globally Lipschitzian and there is a number \( a > 0 \) such that \( |\nabla G(x)| < a|x| \) for all \( x \in \mathbb{R}^n \). Therefore, if we define an operator \( N \) on \( S \) by \( N(u)(t) = \nabla G(u(t)) \) for \( u \in S \) and \( t \in [0, 2\pi] \) the operator \( N \) will map \( S \) continuously into itself. We define \( M : S \to S \) by \( Mu(t) = p(t, u(t)) \) for \( u \in S \) and \( 0 < t < 2\pi \). It is shown in [4, p. 22] that the continuity and boundedness of \( P \) implies that \( M \) is continuous and maps \( S \) into itself. Any \( 2\pi \)-periodic solution of (I) is then a solution of the equation

\[
Lx + Nx = Mx \tag{2.1}
\]

in \( S \), and conversely.

Let \( c = (p + q)/2 \); then \( -c \in \rho(L) \) (the spectrum of \( L \) is \( \{-n^2 : n \in \mathbb{Z}\} \)) and \( (L + cl)^{-1} \) exists as a bounded linear operator on \( S \) and it follows from known results concerning Green's functions (generalized to the case of
uncoupled equations in $\mathbb{R}^n$) that $(L + cI)^{-1}$ is compact [2, p. 192]. Equation (2.1) is equivalent to

$$x = (L + cI)^{-1}[M + cI - N]x.$$  

(2.2)

We will use the Schauder fixed point theorem [11, p. 25] to show that (2.2) has a solution. The operator $M + cI - N$ is continuous and maps bounded sets in $S$ into bounded sets. The compactness of $(L + cI)^{-1}$ now implies that the operator $T$ defined by $T = (L + cI)^{-1}(M + cI - N)$ is completely continuous. If we can show that $T$ maps some closed ball in $S$ into itself we are done.

We first compute \(||(L + cI)^{-1}||\). Because $L$ is selfadjoint and the interval $(-\tilde{n} + 1, -\tilde{n}^2)$ contains no numbers in the spectrum of $L$ we have as in [9]:

\[
||(L + cI)^{-1}|| = \max\{(c - \tilde{n}^2)^{-1}, ((\tilde{n} + 1)^2 - c)^{-1}\}
\]

\[
= \min\{\ldots\}^{-1}.
\]

Now let $u \in S$. We estimate $||(M + cI - N)u||$. Let $H(a) = (\partial^2 G(a)/\partial x_i \partial x_j)$ and $m$ a number with $m > |p(t, x)|$ for all $t, x$. We have

\[
||(M + cI - N)u|| < \|Mu\| + \|(N - cI)u\|
\]

\[
< \sqrt{2\pi} m + \|(N - cI)u\|. \quad (2.4)
\]

Further:

\[
\|Nu - cu\|^2 = \int_0^{2\pi} |\text{grad } G(u(t)) - cu(t)|^2 dt
\]

\[
= \int_0^{2\pi} \left[ \int_0^1 (H(\lambda u(t)) - cI) \, d\lambda \right] |u(t)|^2 dt
\]

\[
< \int_0^{2\pi} \left[ \int_0^1 |H(\lambda u(t)) - cI| \, d\lambda \right] |u(t)|^2 dt
\]

by Taylor’s theorem and since grad $G(0) = 0$.

We make the observation that whenever $|\lambda u(t)| > r$ we have as in [9] by the symmetry of the matrix $(\partial^2 G(\lambda u(t))/\partial x_i \partial x_j) = H(\lambda u(t))$:

\[
|H(\lambda u(t)) - cI| = \sup_{|y|=1} \langle H(\lambda u(t))y - cy, y \rangle
\]

\[
< \max\{(q - c), (c - p)\} = \beta. \quad (2.5)
\]

Let $z(\lambda u(t)) = |H(\lambda u(t)) - cI|$ and choose $\varepsilon > 0$. Let $E_1 = \{(\lambda, t): |\lambda u(t)| < r$ and $0 < \lambda < \varepsilon\}$, $E_2 = \{(\lambda, t): |\lambda u(t)| < r$ and $\varepsilon < \lambda < 1\}$, and $E_3 = \{(\lambda, t): |\lambda u(t)| > r\}$. We have:
\[ \| Nu - cu \|^2 < \int_0^{2\pi} \left( \int_0^1 z(\lambda u(t)) \lambda \|u(t)\|^2 \, dt \right)^2 \]
\[ = \int \left[ \int_{E_1} z(\lambda u(t)) \lambda \|u(t)\|^2 \, dt \right] \]
\[ + \int \left[ \int_{E_2} z(\lambda u(t)) \lambda \|u(t)\|^2 \, dt \right] \]
\[ + \int \left[ \int_{E_3} z(\lambda u(t)) \lambda \|u(t)\|^2 \, dt \right] \]
\[ < e^2 k^2 \|u\|^2 + k^2 \int_{\{t: |u(t)| < r/\epsilon \}} |u(t)|^2 \, dt + \beta^2 \|u\|^2 \]
\[ \leq e^2 k^2 \|u\|^2 + k^2 r^2 / \epsilon^2 + \beta^2 \|u\|^2 \] (2.6)
where \( k = \text{sup}\{ |H(x) - cI| : |x| < r \} \). If \( \|u\| > kr/\epsilon^2 \) we have from (2.6):
\[ \| Nu - cu \|^2 < e^2 (k^2 + 1) \|u\|^2 + \beta^2 \|u\|^2, \]
\[ \| Nu - cu \|^2 < e^2 (k^2 + 1) + \beta^2 \|u\|. \] (2.7)

Since \( \beta = \max\{(q - c), (c - p)\} \) and
\[ \|(L + cI)^{-1}\| = \left[ \min\{ \ldots \} \right]^{-1} \]
and \( c = (p + q)/2 \) with \( \bar{n}^2 < p < q < (\bar{n} + 1)^2 \) it is clear that
\[ \beta \|(L + cI)^{-1}\| < 1. \] (2.8)

Thus we may choose \( \epsilon = \epsilon_1 \) so small that whenever \( \|u\| > kr/\epsilon^2 \) we have by (2.7) and (2.8) that there is a number \( d, 0 < d < 1, \) such that
\[ \|(L + cI)^{-1}\| \| Nu - cu \| < d \|u\|. \] (2.9)

We can now show that the operator \( T = (L + cI)^{-1}(M + cI - N) \) maps a closed ball of the form \( B_n = \{ u \in S : \|u\| < n \}, n \) a positive integer, into itself. If not, then we can find a sequence \( \{ x_n \} \in S \) with \( x_n \in B_n \) and \( \| Tx_n \| > n \). The sequence \( \{ x_n \} \) must tend to infinity in norm, since otherwise \( T \) would be mapping a bounded set onto an unbounded one. Hence \( \|x_n\| \to \infty \) and \( \|x_n\| > rk/\epsilon_1 \) for all but finitely many \( n \). By (2.4) and (2.9) we have for large \( n \):
\[ \|x_n\| < n < \| Tx_n \| < d \|x_n\| + \sqrt{2\pi} \ m \|(L + cI)^{-1}\| \]
and
\[ \|x_n\| < (1 - d)^{-1} \sqrt{2\pi} \ m \|(L + cI)^{-1}\|. \]
This contradicts the unboundedness of the \( \{ x_n \} \). Hence \( T \) maps some ball \( B_k \) into itself. By the Schauder theorem there exists \( x_0 \in B_k \) with \( x_0 = Tx_0 \). Hence \( x_0 \in D(L) \) and
\[ Lx_0 = -Nx_0 + Mx_0. \]

Since \( x_0(t) \) is continuous on \([0, 2\pi]\) and
\[ x''_0(t) = -\nabla G(x_0(t)) + p(t, x_0(t)) \]
it follows that \( x''_0(t) \) is continuous on \([0, 2\pi]\). The function \( x_0(t) \) may now be extended periodically to all of \( R \). This extension is clearly a periodic solution of (I) on all of \( R \). This completes the proof of the theorem.

**Remark 1.** The uniform boundedness of \( p(t, x) \) was not essential, and it is clear that the theorem remains true if \( p(t, x) \) is sublinear, i.e., if \( \lim_{|x| \to \infty} |p(t, x)|/|x| = 0 \), as \( |x| \to \infty \), the convergence being uniform in \( t \).

**Remark 2.** This method is not restricted to the periodic problem, but could also be used to handle other problems.

**Remark 3.** Reissig [10] has extended Mawhin's approach to the equation
\[ x'' + Cx' + \nabla G(x) = e(t) \]
with \( C \) symmetric. His results can be combined with the methods of this paper to give similar results to our Theorem 1 for the equation
\[ x'' + Cx' + \nabla G(x) = p(t, x). \]

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**References**


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