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THE BANACH-MAZUR DISTANCE BETWEEN THE TRACE CLASSES c_p^n

NICOLE TOMCZAK-JAEGERMANN

ABSTRACT. The Banach-Mazur distance between $l_2^n \hat{\otimes} l_2^m$ and $l_2^n \hat{\hat{\otimes}} l_2^m$ is shown to be of the order $\sqrt{\min(n,m)}$. Our proof yields that the distance between the trace classes c_p^n and c_q^n is of the same order as $d(l_p^n, l_q^n)$.

In this note we determine the distances between some tensor products of Euclidean spaces l_2^k, $k = 1, 2 \ldots$. Let E, F be finite dimensional Banach spaces over the real field. The Banach-Mazur distance $d(E, F)$ is defined as

$$\inf\{\|T\|\,\|T^{-1}\| \mid T \text{ is an isomorphism from } E \text{ onto } F\}.$$

In this note by $E \hat{\otimes} F$ [resp. $E \hat{\hat{\otimes}} F$] we denote the algebraic tensor product $E \otimes F$ endowed with the greatest [resp. the least] norm such that $\|e \otimes f\| = \|e\|\,\|f\|$ for $e \in E, f \in F$. The space $l_2^n \otimes l_2^m$ with the norm

$$\left\|\sum_{ij} a_{ij} e_i \otimes f_j\right\| = \left(\sum_{ij} |a_{ij}|^2\right)^{1/2},$$

where $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_m\}$ are orthonormal bases for l_2^n and l_2^m respectively, is denoted by $\mathrm{HS}(l_2^n, l_2^m)$ or simply HS.

THEOREM 1. *Let n, m be positive integers with $n \leqslant m$. Then*

$$(2\sqrt{e})^{-1}\sqrt{n} \leqslant d\left(l_2^n \hat{\otimes} l_2^m, l_2^n \hat{\hat{\otimes}} l_2^m\right) \leqslant 10\sqrt{n}.$$

PROOF. We begin with the upper estimate of the distance $d(l_2^n \hat{\otimes} l_2^m, l_2^n \hat{\hat{\otimes}} l_2^m)$. The argument given below works only for $n \geqslant 36$. However if $n < 36$ and i: $l_2^n \hat{\otimes} l_2^m \to l_2^n \hat{\hat{\otimes}} l_2^m$ denotes the formal identity map, then $\|i\| \leqslant 1$ and $\|i^{-1}\| \leqslant n \leqslant 10\sqrt{n}$.

We shall construct the isomorphism T: $l_2^n \hat{\otimes} l_2^m \to l_2^n \hat{\hat{\otimes}} l_2^m$ in the form $T = j^* \circ u \circ j$ where j: $l_2^n \hat{\otimes} l_2^m \to \mathrm{HS}(l_2^n, l_2^m)$ is the natural embedding and u is an isometry of the nm-dimensional Hilbert space $\mathrm{HS}(l_2^n, l_2^m)$. It is easy to check that $\|j^{-1}\| \leqslant \sqrt{n}$. Since $(j^*)^{-1} = (j^{-1})^*$ we obtain

$$\|T^{-1}\| \leqslant \|j^{-1}\| \cdot \|u^{-1}\| \cdot \|(j^*)^{-1}\| \leqslant n.$$

Thus the proof of the upper estimate will be complete if we find a T with $\|T\| \leqslant 10/\sqrt{n}$. This is done in the following proposition.

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PROPOSITION. *Let μ denote the normalized Haar measure on the group G of all linear isometries of* $\mathrm{HS}(l_2^n, l_2^m)$. *Assume that $m \geqslant n \geqslant 36$. Then*

$$\mu\{u \in G \mid \|j^* \circ u \circ j\| > 10/\sqrt{n}\} < 1.$$

PROOF. Observe that the set E of the extreme points of the unit ball in $l_2^n \hat{\otimes} l_2^m$ equals $S_{n-1} \times S_{m-1}$ where S_{k-1} denotes the unit sphere in l_2^k. It follows from the duality between $l_2^n \hat{\otimes} l_2^m$ and $l_2^n \hat{\hat{\otimes}} l_2^m$ that

$$\|j^* \circ u \circ j\| = \sup\{|\langle j(z \otimes w), u(j(x \otimes y))\rangle| \mid (x, y), (z, w) \in E\},$$

where $\langle \cdot\,, \cdot \rangle$ denotes the inner product in the space $\mathrm{HS}(l_2^n, l_2^m)$. The latter expression is equal to the norm of the 4-linear form $\tilde{u}$ on $l_2^n \times l_2^m \times l_2^n \times l_2^m$ defined by

$$\tilde{u}(x, y, z, w) = \langle j(z \otimes w), u(j(x \otimes y))\rangle.$$

Pick 1/8-nets N_1 in S_{n-1} and N_2 in S_{m-1} and let $N = N_1 \times N_2 \times N_1 \times N_2$. We may assume that $\operatorname{card} N_1 \leqslant e^{3n}$ and $\operatorname{card} N_2 \leqslant e^{3m}$. (This can be proved by a standard comparison of volumes argument, cf. e.g. [**1**, Lemma 2.4].) Thus $\operatorname{card} N \leqslant e^{6nm} \leqslant e^{12m}$. Given $(x, y, z, w) \in E \times E$ pick $\xi = (x', y', z', w') \in N$ such that $\|x - x'\|, \|y - y'\|, \|z - z'\|, \|w - w'\| \leqslant 1/8$. One can check easily that

$$|\tilde{u}(x, y, z, w) - \tilde{u}(x', y', z', w')| \leqslant \left(\tfrac{4}{8}\right)\|\tilde{u}\|,$$

and hence

$$\|\tilde{u}\| \leqslant \sup_{\xi \in N} |\tilde{u}(\xi)| + \tfrac{1}{2}\|\tilde{u}\|,$$

so that

$$\|\tilde{u}\| \leqslant 2 \sup_{\xi \in N} |\tilde{u}(\xi)|.$$

Observe that

$$\begin{aligned}\{u \in G \mid \|\tilde{u}\| > 10/\sqrt{n}\} &\subseteq \left\{u \in G \mid \sup_{\xi \in N} |\tilde{u}(\xi)| > 5/\sqrt{n}\right\} \\ &\subseteq \bigcup_{\xi \in N} \{u \in G \mid |\tilde{u}(\xi)| > 5/\sqrt{n}\}.\end{aligned}$$

We shall use the estimate

$$v_\xi = \mu\{u \in G \mid |\tilde{u}(\xi)| > a\} \leqslant 4 \exp(-a^2 \cdot nm/2)$$

valid for any $\xi \in E \times E$ and $0 < a \leqslant 5/6$.

Let $a = 5/\sqrt{n}$. Since $n \geqslant 36$, we have $a \leqslant 5/6$ and hence

$$\begin{aligned}\mu\{u \in G \mid \|\tilde{u}\| > 10/\sqrt{n}\} &\leqslant \operatorname{card} N \cdot v_\xi \leqslant 4e^{12m}e^{-25m/2} \\ &= 4e^{-m/2} \leqslant 4e^{-18} < 1,\end{aligned}$$

which is the desired estimate.

To estimate v_ξ observe that

$$v_\xi = \lambda\{t \in S_{nm-1} \mid |\langle j(z \otimes w), t\rangle| > a\},$$

where λ denotes the $(nm-1)$-dimensional normalized Lebesgue measure on the sphere S_{nm-1}. By the formula 2.4 in [1] the latter number is smaller than

$$4\exp\left(-(\sin^{-1}a)^2 nm/2\right) \leqslant 4\exp(-a^2 nm/2).$$

(The estimate in [1] requires that $\sin^{-1}a < 1$, but in our case $a \leqslant 1 - 1/6 < \sin 1$.) This completes the proof of the proposition.

The lower estimate of $d(l_2^n \hat{\otimes} l_2^m, l_2^n \hat{\hat{\otimes}} l_2^m)$ is obvious if one notes that the cotype 2 constant (cf. e.g. [1]) of the first space is $\leqslant 2\sqrt{e}$ (cf. [3]) while the second space contains an isometric copy of l_∞^n, hence its cotype 2 constant is $\geqslant \sqrt{n}$. This completes the proof of Theorem 1.

In the next theorem c_p^n denotes the tensor product $l_2^n \otimes l_2^n$ with the norm

$$\|u\|_p = \left(\operatorname{trace}(u^* r)^{p/2}\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty$$

and

$$\|u\|_\infty = \|u\|,$$

where $u \in l_2^n \otimes l_2^n$ is regarded as a linear operator in l_2^n. Thus c_2^n may be identified with $\mathrm{HS}(l_2^n, l_2^n)$.

THEOREM 2. *For arbitrary* $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ *one has*

$$(2\sqrt{e})^{-1} n^\alpha \leqslant d(c_p^n, c_q^n) \leqslant 10 n^\alpha,$$

where $\alpha = \max(1/p - 1/2, 1/2 - 1/q)$.

PROOF. Assume that $1 \leqslant q^* \leqslant p \leqslant 2$, where $q^* = (p-1)/p$. Then $\alpha = 1/2 - 1/q$. We shall prove that $d(c_{p^*}^n, c_q^n) \leqslant 10^{2/p-1} n^\alpha$ and $d(c_p^n, c_q^n) \geqslant (2\sqrt{e})^{-1} n^\alpha$. The remaining cases follow from the identity

$$d(c_p^n, c_q^n) = d((c_p^n)^*, (c_q^n)^*) = d(c_{p^*}^n, c_{q^*}^n)$$

and the observation that $2/p - 1 \leqslant 1$.

Let us estimate the norm $\|T\colon c_{q^*}^n \to c_{p^*}^n\|$, where T is the operator we have used in the proof of Theorem 1. We have

$$\begin{aligned}\|T\colon c_{q^*}^n \to c_{p^*}^n\| &\leqslant \|T\colon c_p^n \to c_{p^*}^n\| \\ &\leqslant \|T\colon c_1^n \to c_\infty^n\|^{2/p-1} \|T\colon c_2^n \to c_2^n\|^{2-2/p} \\ &\leqslant (10 n^{-1/2})^{2/p-1} = 10^{2/p-1} n^{1/2-1/p}.\end{aligned}$$

The interpolation theorem we have used in the last estimate can be found e.g. in [3].

On the other hand

$$\begin{aligned}\|T^{-1}\colon c_{p^*}^n \to c_{q^*}^n\| &\leqslant \|\mathrm{id}\colon c_{p^*}^n \to c_2^n\| \cdot \|u^{-1}\colon c_2^n \to c_2^n\| \cdot \|\mathrm{id}\colon c_2^n \to c_{q^*}^n\| \\ &\leqslant n^{1/p-1/2} \cdot 1 \cdot n^{1/2-1/q}.\end{aligned}$$

Thus we get

$$d(c_{p^*}^n, c_{q^*}^n) \leqslant \|T\| \cdot \|T^{-1}\| \leqslant 10^{2/p-1} n^{\alpha}.$$

The lower estimate of $d(c_p^n, c_q^n)$ uses the facts that the cotype 2 constant of c_p^n is $\leqslant 2\sqrt{e}$ (cf. [3]) and the cotype 2 constant of c_q^n is $\geqslant n^{\alpha}$. (The latter space contains an isometric copy of l_q^n.)

REMARK. It is well known that if either $1 \leqslant p, q \leqslant 2$ or $2 \leqslant p, q \leqslant \infty$, then

$$d(c_p^n, c_q^n) = n^{\beta} \quad \text{where } \beta = |1/p - 1/q|.$$

The result mentioned in the abstract follows by comparing our results with those in [2].

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INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, WARSAW, POLAND

Current address: Institute of Mathematics, Warsaw University, PKiN, 00901 Warsaw, Poland