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THE MILNOR NUMBER OF SOME ISOLATED COMPLETE INTERSECTION SINGULARITIES WITH C*-ACTION

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ABSTRACT. We compute the Milnor number of isolated complete intersection singularities defined by weighted homogeneous polynomials of the same type. We use this result to compute the characteristic polynomial of a certain monodromy and thus obtain some information about the link of the singularity. We also discuss the question of when such a complete intersection with specified weights exists.

Introduction. We will consider complete intersections V^n in $\mathbf{C}^m$ which are defined by weighted homogeneous polynomials. Recall that a polynomial $f(z_1, \ldots, z_m)$ is said to be weighted homogeneous with weights $(w_1, \ldots, w_m)$, $w_j \in Q$ and $w_j > 1$, if for every monomial $\alpha z_1^{a_1} \ldots z_m^{a_m}$ one has

$$a_1/w_1 + \cdots + a_m/w_m = 1. \tag{1}$$

Let $V(f) = \{\underline{z} \in \mathbf{C}^m | f(\underline{z}) = 0\}$. Then $V(f)$ is invariant under an action of $\mathbf{C}^* = \mathbf{C} - \{0\}$ given by

$$\sigma(t; z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m), \tag{2}$$

where the exponents $q_1, \ldots, q_m$ are obtained by writing $w_j = u_j/v_j$, setting $d = \operatorname{lcm}(u_1, \ldots, u_m)$, and defining $q_j = d/w_j$. Such $\mathbf{C}^*$-varieties were introduced by Milnor [8].

Here we wish to consider more general varieties $V = V(f^1, \ldots, f^k) = V(f^1) \cap \ldots \cap V(f^k) \subset \mathbf{C}^m$, where (i) each f^i is a weighted homogeneous polynomial with weights $(w_1, \ldots, w_m)$ independent of i, and (ii) for all $s = 1, \ldots, k$, $V(f^1, \ldots, f^s)$ is a complete intersection with an isolated singularity at the origin in $\mathbf{C}^m$. We will write $V \in V^n(w_1, \ldots, w_m)$, where $n = m - k$, when (i) and (ii) are satisfied.

Observe that (i) implies that V is invariant under a $\mathbf{C}^*$-action as in (2). If we let f: $\mathbf{C}^m \to \mathbf{C}^k$ be the holomorphic function with coordinates $f^1, \ldots, f^k$, then (ii) implies that the $k \times m$ Jacobian matrix $(\partial f^i/\partial z_j)$ has rank k everywhere in some neighborhood of the origin in V except possibly at the

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origin itself. If $V(f^1, \ldots, f^k)$ is a complete intersection with isolated singularity, then by [2, Lemma 2.2] one may make a linear coordinate change in $\mathbf{C}^k$ so that (ii) holds.

Later we will make further comments about $V^n(w_1, \ldots, w_m)$ but now let us state our main result. It has been shown by Milnor [8] if $k = 1$ and Hamm [4] for $k > 1$ that there is a fundamental fibration associated to isolated singularities of complete intersections. Namely, let D and S be suitably small neighborhoods of $\underline{0}$ in $\mathbf{C}^m$ and $\mathbf{C}^k$ respectively, let Δ be the set of critical values in S of the restriction of f to D, and let $S' = S - \Delta$ and $D' = D - f^{-1}(\Delta)$. Then $f\colon D' \to S'$ is a smooth fiber bundle. The fiber F is a smooth open parallelizable $2n$-manifold which has the homotopy type of a wedge (or bouquet) of n-spheres. The number of n-spheres is denoted $\mu(V)$ and called the Milnor number (or multiplicity) of the singularity.

THEOREM 1. *Suppose* $V \in V^n(w_1, \ldots, w_m)$. *Let* $p(t) = \prod_{j=1}^m (w_j t + (w_j - 1)) = \beta_m t^m + \cdots + \beta_1 t + \beta_0$. *Then*

$$\mu(V) = \beta_{k-1} - \beta_{k-2} + \cdots + (-1)^{k-1}\beta_0. \tag{3}$$

In §1 we prove this formula for μ. Then in §2 we use it to compute an associated characteristic polynomial. This is used in turn to obtain bounds on the rank of the homology of the link of the singularity. Finally, we briefly consider the question of when $V^n(w_1, \ldots, w_n)$ is nonempty.

After completing this work it was learned that M. Giusti, G.-M. Greuel and H. Hamm have obtained similar results on μ; see Greuel's announcement [3].

1. The Milnor number. In this section we will prove Theorem 1, but before we do so, let us note that it implies that μ depends only on $(n; w_1, \ldots, w_m)$. In fact, one may show that any two varieties in $V^n(w_1, \ldots, w_m)$ have diffeomorphic Milnor fibers.

We also note that Theorem 1 gives an integrality condition: If $\beta_{k-1} - \cdots + (-1)^{k-1}\beta_0$ is not an integer, then $V^n(w_1, \ldots, w_m)$ is empty. It will follow from the proof that in fact each β_i, $i = 0, \ldots, k-1$, must be an integer.

PROOF OF THEOREM 1. Let $R = \mathbf{C}[z_1, \ldots, z_m]$, and let $(T_1, \ldots, T_r)$ denote the ideal in R generated by the polynomials or ideals $T_1, \ldots, T_r$. Given an ideal I of R, we will let $V(I)$ (resp. $C(I)$) denote the variety (resp. cycle) defined by I. Similarly, given a variety V in $\mathbf{C}^m$, we write $I(V)$ for the ideal (in R) defined by V.

We will use the following formula, due independently to Greuel [2] and Lê [7]. Let J_s be the ideal of R generated by all minors of size s of the $s \times m$ matrix $(\partial f^i / \partial z_j)$, $i = 1, \ldots, s$; $j = 1, \ldots, m$. Let $m^s = \dim_{\mathbf{C}} R/(f^1, \ldots, f^{s-1}, J_s)$. Then

$$\mu\big(V(f^1, \ldots, f^k)\big) = m^k - m^{k-1} + \cdots + (-1)^{k-1} m^1. \tag{4}$$

Thus we must show $m^s = \beta_{s-1}$, $s = 1, \ldots, k$.

To accomplish this we lift the problem to one involving homogeneous polynomials, interpret m^s as an intersection number, and deform the intersecting cycles in $\mathbf{P}^m$ so that this intersection number is easily calculated.

First, define φ: $\mathbf{C}_0^m \to \mathbf{C}^m$ by $\varphi(z_1, \ldots, z_m) = (z_1^{q_1}, \ldots, z_m^{q_m})$. We will use the subscript 0 to denote objects associated with the domain of φ. If V is a variety in $\mathbf{C}^m$ left invariant by an action of the form of (2), V defined by $f^i(z_1, \ldots, z_m)$, then $V_0 = \varphi^{-1}(V)$ is defined by the homogeneous polynomials $f_0^i = f^i \circ \varphi = f(z_1^{q_1}, \ldots, z_m^{q_m})$.

Now m^s is just the local intersection multiplicity at 0 of $C(f^1), \ldots, C(f^{s-1}), C(J_s)$. Letting m_0^s be the local intersection multiplicity at 0 in $\mathbf{C}_0^m$ of $C(f_0^1), \ldots, C(f_0^{s-1}), C(\varphi^{-1}(j_s))$, we have $q_1 \cdots q_m \cdot m^s = m_0^s$ by an easy direct argument, as in [**1**, Proposition 4.3].

Next we take the canonical embedding of $\mathbf{C}_0^m \subset \mathbf{P}_0^m$. For any i, the polynomial $(\partial f^i/\partial z_j) \circ \varphi$ has degree $d - q_j$. Thus it is easily seen that $\varphi^{-1}(J_s)$ is generated by homogeneous polynomials. Therefore, since $V(f_0^1), \ldots, V(f_0^{s-1}), V(\varphi^{-1}(J_s))$ intersect only at 0 in $\mathbf{C}_0^m$, they also intersect only at this point in $\overline{\mathbf{C}}_0^m = \mathbf{P}_0^m$. Thus m_0^s is the *global* intersection multiplicity of $C(f_0^1), \ldots, C(f_0^{s-1}), C(\varphi^{-1}(J_s))$ in $\mathbf{P}_0^m$.

Now we deform $C(\varphi^{-1}(J_s))$ to an algebraically equivalent cycle C_0, defined as follows. Let $M = (m_{ij})$ be any $s \times m$ matrix over $\mathbf{C}$ such that no minor is zero. Let $M(f_0^1)$ be the $s \times m$ matrix $(m_{ij}(\partial f^1/\partial z_j) \circ \varphi)$. Let J_0 be the ideal generated by all $s \times m$ minors of $M(f_0^1)$, and let $C_0 = C(J_0)$.

For some integer K we may identify $\mathbf{C}^K$ with the space of all $s \times m$ matrices with ijth entry a polynomial of degree $d - q_j$. Let $T \subset \mathbf{C}^K$ be the set of all $t \in \mathbf{C}^K$ so that the variety $V_0(t)$ defined by the vanishing of all $s \times s$ minors of the corresponding matrix has dimension $s - 1$. Let $C_0(t)$ be the corresponding cycle. Since $\dim V(t) \geqslant s - 1$ for all t, and since strict inequality is an algebraic condition, T is a Zariski open subvariety of $\mathbf{C}^K$. Define a cycle Z_1 as the fiber of the map ψ: $\mathbf{C}_0^m \times T \to \mathbf{C}^r$, where $\psi(z, t) = (m_1(z, t), \ldots, m_r(z, t))$ with $m_i(\cdot, t)$ the minors of size s of the matrix corresponding to t. Let Z be the closure of Z_1 in $\mathbf{P}_0^m \times T$. Alternatively, one may define Z by the ideal generated by the $m_i(\underline{z}, t)$.

From the definition of T, it follows that Z and $\mathbf{P}_0^m \times t$ intersect properly for any $t \in T$. Since $C(\varphi^{-1}(J_s)) = Z \cdot (\mathbf{P}_0^m \times t_1)$ and $C_0 = Z \cdot (\mathbf{P}_0^m \times t_2)$, we will have shown that Z is an algebraic equivalence between these two cycles once we show that $t_1, t_2 \in T$. But since m^s is finite, we know $\dim V(J_s) = \dim V(\varphi^{-1}(J_s)) = s - 1$, and so $t_1 \in T$.

We will show $t_2 \in T$ in the course of proving a stronger statement. Let $g_j = (\partial f^1/\partial f_j) \circ \varphi$, and let I_α be the ideal in $\mathbf{C}[z_1, \ldots, z_m]$ generated by $g_{j_1}, \ldots, g_{j_{m-s+1}}$, where $\alpha = \{j_1, \ldots, j_{m-s+1}\}$ runs over all $(m - s + 1)$-element subsets of $\{1, \ldots, m\}$. We show $C_0 = \sum_\alpha C(I_\alpha)$. We work in the open subset $\mathbf{C}_0^m$ of $\mathbf{P}_0^m$. First, as sets we have $V(J_0) = \bigcup_\alpha V(I_\alpha)$, since J_0 is generated by all possible products of s of the g_j, and all such products vanish

precisely where $m - s + 1$ of the g_j vanish. But since f^1 has an isolated singularity at $\underline{0}$, the set $(g_1, \ldots, g_m)$ defines a complete intersection, and thus so does any subset. Thus $\dim V(I_\alpha) = s - 1$ for all α, and $t_2 \in T$.

Finally, to obtain the cycle equation we take an irreducible component X of $V(J_0)$. Let p be the prime ideal of polynomials vanishing on X, and let $(\ )_p$ denote localization at p. Letting $I = \prod_\alpha I_\alpha$, we have $C(I) = \sum_\alpha C(I_\alpha)$. To show that X has the same multiplicity on both sides of the cycle equation we must show $(J_0)_p = I_p$. But $X \subset V(I_\alpha)$ for precisely one α, since otherwise $m - s + 2$ partials of f^1 would vanish on $\varphi(X)$, implying $\dim \varphi(X) \leqslant s - 2$, a contradiction. Take $\alpha = \{1, \ldots, m - s + 1\}$ for convenience. Thus by the above, $g_j \notin p$, for $j > m - s + 1$. Since J_0 is generated by all products of s of the g_j, we have

$$g_j g_{m-s+2} g_{m-s+3} \cdots g_m / g_{m-s+2} g_{m-s+3} \cdots g_m = g_j/1 \in (J_0)_p,$$

for all $j \leqslant m - s + 1$, and thus $(J_0)_p = (g_1, \ldots, g_{m-s+1})_p$. A similar argument shows $I_p = (g_1, \ldots, g_{m-s+1})_p$, giving the desired equality of cycles.

Now we have the global intersection number

$$\begin{aligned} m_0^s &= C(f_0^1) \cdot \ldots \cdot C(f_0^{s-1}) \cdot C(\varphi^{-1}(J_s)) \\ &= C(f_0^1) \cdot \ldots \cdot C(f_0^{s-1}) \cdot C_0, \quad \text{by the equivalence} \\ &= C(f_0^1) \cdot \ldots \cdot C(f_0^{s-1}) \cdot \sum_\alpha C_\alpha \\ &= \sum_\alpha d^{s-1}(d - q_{j_1}) \ldots (d - q_{j_{m-s+1}}), \quad \text{by the ordinary Bezout Theorem.} \end{aligned}$$

This equality and the equation $q_1 \ldots q_m m^s = m_0^s$ show $m^s = \beta_{s-1}$, and thus complete the proof.

REMARK. One sometimes considers polynomials with different weights which have the same $(q_1, \ldots, q_m)$ (i.e. the d's depend on the polynomials). The above proof fails to generalize to this situation because $\varphi^{-1}(J_s)$ is not in general a homogeneous ideal and $Z \cap (\mathbf{P}^m \times t_1)$ may be strictly bigger than $\mathrm{cl}(V(\varphi^{-1}(J_s))) \subset \mathbf{P}^m$.

2. The characteristic polynomial of a monodromy operator. We next use Theorem 1 to compute the characteristic polynomial of a certain monodromy. Consider $V(f^1, \ldots, f^k) \subset V(f^1, \ldots, f^{k-1})$. By intersecting with a sufficiently small sphere centered at $\underline{0}$ in $\mathbf{C}^m$ we obtain a pair of manifolds $K \subset K^*$ of dimension $2n - 1$ and $2n + 1$. Hamm [**4**] shows that $f^k/|f^k|$: $K^* - K \to S^1$ is a smooth fiber bundle with fiber F (same F as before). As usual, $K^* - K$ is formed from $F \times [0, 1]$ by identifying the ends via a diffeomorphism h: $F \to F$. Let H be the matrix of the map h_*: $H_n(F) \to H_n(F)$. Then $\Delta(t) = \mathrm{Det}(tI - H)$ gives information about $K^* - K$ and thus about K. Theorem 1 allows us to give the following generalization of [**9**, Theorem 4] and [**5**, Lemma 4.2]. We use the divisor notation of the former paper.

THEOREM 2. *Suppose* $V \in V^n(w_1, \ldots, w_m)$. *Then*

Divisor $\Delta_V(t)$

$$= \sum_{r=0}^{k-1} \sum_{I_r} (-1)^{r-k+1} \frac{1}{v_{\sigma 1}} \Lambda_{u_{\sigma 1}} \cdots \frac{1}{v_{\sigma r}} \Lambda_{\sigma r}\left(\frac{1}{v_{\sigma r+1}} \Lambda_{u_{\sigma r+1}} - 1\right) \cdots \left(\frac{1}{v_{\sigma m}} \Lambda_{u_{\sigma m}} - 1\right) \quad (5)$$

where I_r *runs over all partitions of* $\{1, \ldots, m\}$ *into two sets* $\{\sigma 1, \ldots, \sigma r\}$, $\{\sigma r + 1, \ldots, \sigma m\}$ *of* r *and* $m - r$ *elements respectively.*

PROOF. The proof is like that of Theorem 4 in [**9**]. The diffeomorphism h may be taken with finite order, as in [**8**]. Then the fixed point sets of various iterates of h are formed by letting various subsets of the variables be zero. These fixed-point sets are then themselves fibers corresponding to isolated singularities of complete intersections defined by weighted homogeneous polynomials, and the result follows from our Theorem 1.

As mentioned, $\Delta(t)$ gives some information about $K^* - K$, K^*, and K. Specifically there are exact sequences

$$0 \to H_{n+1}(K^* - K) \to H_n(F) \xrightarrow{I - h_*} H_n(F) \to H_n(K^* - K) \to 0 \quad (6)$$

and

$$\begin{aligned} 0 \to \tilde{H}^{n-1}(K) \to H_{n+1}(K^* - K) \to H_{n+1}(K^*) \\ \to H^n(K) \to H_n(K^* - K) \to H_n(K^*) \to 0. \end{aligned} \quad (7)$$

The sequence (6) is a Wang sequence, while (7) follows from the homology sequence of $(K^*, K^* - K)$ together with Alexander duality.

From (6) and (7) one obtains Hamm's results that $\Delta(1)$ is a unit in $\mathbf{Z}$ or Q if and only if K and K^* are $\mathbf{Z}$ or Q homology spheres, respectively. Here is a more general result which places computable bounds on the betti numbers of K. Let $\kappa(K) = \operatorname{rank} \tilde{H}_{n-1}(K)$, $\kappa(K^*) = \operatorname{rank} H_n(K^*)$, $\kappa(h_*) = \operatorname{rank} \operatorname{coker}(I - h_*)$, and $\kappa(h_*^*) = \operatorname{rank} \operatorname{coker}(I - h_*^*)$, where h_*^* is the monodromy associated to $V(f^1, \ldots, f^{k-1}) \subset V(f^1, \ldots, f^{k-2})$. Then $\kappa(h_*)$ and $\kappa(h_*^*)$ are explicitly computable for $V \in V^n(w_1, \ldots, w_m)$, and we have

THEOREM 3. *Suppose* $V(f^1, \ldots, f^s)$ *is a complete intersection with isolated singularity at* $\underline{0}$, *for* $s = 1, \ldots, k$. *Then*

$$\kappa(h_*) - \kappa(h_*^*) \leqslant \kappa(h_*) - \kappa(K^*) \leqslant \kappa(K) \leqslant \kappa(h_*). \quad (8)$$

PROOF. From (6) we obtain $\kappa(h_*) = \operatorname{Rank} H_{n+1}(K^* - K)$, and this and (7) yield $\kappa(K) \leqslant \kappa(h_*)$. Similarly $\kappa(K^*) \leqslant \kappa(h_*^*)$, giving the left-hand inequality. Finally, sequence (7) implies $\kappa(h_*) \leqslant \kappa(K) + \kappa(K^*)$, and this gives the middle inequality.

We illustrate these results with a simple example, $V \in V^1(5, 9, 45/4)$. The

defining polynomials are $f^i(z_1, z_2, z_3) = \alpha_{i1}z_1^5 + \alpha_{i2}z_2^9 + \alpha_{i3}z_1z_3^9 + \alpha_{i4}z_2z_3^{10}$, $i = 1, 2$. One easily computes $\mu(V) = 811$, and $\Delta_V(t) = (t^{45} - 1)^{18}(t - 1)$.Furthermore, f^1 defines $V^* \in V^2(5, 9, 45/4)$, and $\mu_{V^*}(t) = 328$ with $\Delta_{V^*}(t) = (t^{45} - 1)^7(t^9 - 1)(t^5 - 1)(t - 1)^{-1}$. The associated K^* and K have dimension three and one, respectively. Finally, $\kappa(h_*^*) = 8$ and $\kappa(h_*) = 19$, so that rank $H_1(K^*) = 8$ and $11 \leqslant$ rank $\tilde{H}_0(K) \leqslant 19$.

The example above shows that there are nonempty $V^n(w_1, \ldots, w_m)$ with $k > 1$ and nonintegral weights. Of course there are also well-known examples for all k with any integral weights, given by Pham-Brieskorn polynomials.

Now consider weights $(3, 9/2, 18/7)$. The polynomial $f(z_1, z_2, z_3) = z_1^3 + z_1z_2^3 + z_2z_3^2$ has these weights, showing that $V^2(3, 9/2, 18/7)$ is nonempty. On the other hand, $V^1(3, 9/2, 18/7)$ is empty. One may see this by plugging these weights and $n = 1$ (i.e., $k = 2$) into the formula of Theorem 1. The result is not an integer.

In fact, the monomials of f above are the only possible ones for these weights. By considering the Jacobian J of general linear combinations f^1 and f^2 of these monomials it is easily seen that Rank $J < 2$ on the z_3-axis, so that $V^1(3, 9/2, 18/7) = \varnothing$. Similar direct considerations show whether a particular $V^n(w_1, \ldots, w_m)$ is nonempty, though the computation may be lengthy.

See [**10**] for a discussion of this in the hypersurface case.

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