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FIXED POINTS OF SEQUENCES OF LOCALLY EXPANSIVE MAPS

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ABSTRACT. A stability theorem for fixed points of a uniformly convergent sequence of open, locally expansive maps is obtained.

1. Introduction. When does a sequence of fixed points a_n of a convergent sequence of functions f_n from a locally compact space X into itself converge to a fixed point a_0 of the limit, f_0? In **[3]**, Fraser and Nadler show it does if the functions are contractive and the convergence is pointwise. Here, we show it does if all the functions are open, locally expansive on a compact connected absolute neighborhood retract X and the convergence is uniform. The idea is easy enough. We lift the sequence and its limit to the universal covering space $\tilde{X}$ of X and then show the inverses of the lifts satisfy a result like Theorem 1 in **[3]** but for locally contractive maps. The author gratefully acknowledges a helpful communication from Professor Paul Duvall in which Theorem 22 of **[1]** is simplified and extended to locally expansive maps.

Let (X, d) be a compact metric space. A continuous function $f\colon X \to X$ is called *locally expansive* (or *locally ε-expansive*) if there is an $\varepsilon > 0$ such that $0 < d(x,y) < \varepsilon$ implies $d(f(x), f(y)) > d(x,y)$. Rosenholtz shows in **[4]** that if such a function f is also open and if X is also connected, then f has a fixed point.

An example of a locally expansive map of the torus $S^1 \times S^1$ onto itself is $f(z, w) = (z^2, w^3)$ where the unit circle S^1 is viewed as a subset of the set of complex numbers. We may think of f mapping meridians twice around themselves and longitudes three times around themselves.

2. Stability of fixed points. Let X be a compact connected ANR (e.g., a closed, connected n-manifold) with metric d, and let $\pi\colon \tilde{X} \to X$ be a covering projection. According to Theorem 1 in **[1]**, there is a complete metric $\tilde{d}$ for $\tilde{X}$ and $\eta > 0$ such that if $\tilde{d}(x,y) < \eta$, then $\tilde{d}(x,y) = d(\pi(x), \pi(y))$.

LEMMA 1. *If $g_i\colon \tilde{X} \to X$ is a continuous function for $i = 0, 1, 2, \ldots$ and if the sequence $\{g_i\}_{i=1}^{\infty}$ converges uniformly to g_0, then there exist lifts $\tilde{g}_i\colon \tilde{X} \to \tilde{X}$ of g_i such that $\{\tilde{g}_i\}_{i=1}^{\infty}$ converges pointwise to $\tilde{g}_0$.*

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PROOF. Define a continuous function $f\colon \tilde{X} \times \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \to X$ by $f(x, t) = g_i(x)$ if $t = 1/i$ but $f(x, t) = g_0(x)$ if $t = 0$. Since X is an ANR, there is a continuous function $F\colon \tilde{X} \times [0, 1/n] \to X$ for some n such that $F = f$ on $\tilde{X} \times \{0, 1/n, 1/(n+1), \dots\}$. By two applications of the lifting theorem, [5, p. 76], there exists a lift $\tilde{g}_0$ of g_0 (i.e., $F(x, 0) = \pi\tilde{g}_0$), and there is a continuous function $F'\colon \tilde{X} \times [0, 1/n] \to \tilde{X}$ such that $F'(x, 0) = \tilde{g}_0$ and $\pi F' = F$. Define $\tilde{g}_i(x) = F'(x, 1/i)$ on $\tilde{X}$ for $i \geqslant n$. Therefore $\tilde{g}_i$ is a lift of g_i and $\{\tilde{g}_i\}_{i=1}^{\infty}$ converges pointwise to $\tilde{g}_0$.

REMARKS. This lemma is false if we only assume $\{g_i\}_{i=1}^{\infty}$ converges pointwise to g_0. Some improvements can be made. If $g_0 = f_0\pi$ and if a_0 is a fixed point of f_0, then according to the lifting theorem, we can require any $\tilde{a}_0$ in $\pi^{-1}(a_0)$ to be a fixed point of a function $\tilde{g}_0$ to which we lift g_0. If f_0 is open and locally expansive, then it follows from Theorem 3 in [2] that $\tilde{g}_0$ has only one fixed point. Since F' is uniformly continuous on each compact subset $K \times [0, 1/n]$, $\{\tilde{g}_i\}_{i=1}^{\infty}$ converges uniformly to $\tilde{g}_0$ on each compact subset K of $\tilde{X}$.

LEMMA 2. *If X is a compact, connected, locally connected space with metric d and if $f_i\colon X \to X$ is an open locally ε-expansive map for $i = 0, 1, 2, \dots,$ then there is an open cover $\{W_\beta\}$ of X evenly covered by f_i for $i = 0, 1, 2, \dots$.*

PROOF. Let $\{W_\beta\}$ be an open cover of X such that each W_β is connected and has diameter $< \varepsilon$. Let i and β be any specific indices, and let r denote the diameter of the open set W_β. We first show that each component C of $f_i^{-1}(W_\beta)$ has diameter $< \varepsilon$. On the contrary, assume some component C has diameter $\geqslant \varepsilon$. Since C is connected, there are points a and b in C such that $r < d(a, b) < \varepsilon$. Therefore $d(f_i(a), f_i(b)) > d(a, b) > r$. That is, $f_i(a) \notin W_\beta$ or $f_i(b) \notin W_\beta$, a contradiction. This shows C has diameter $< \varepsilon$.

Since f_i is a locally ε-expansive map and since each C is an open set, f_i is a 1-1 open function on C. We maintain f_i maps each C onto W_β. Assume otherwise. Since W_β is connected, there is a point y of W_β in the boundary of $f_i(C)$. Corresponding to this β and this i is some neighborhood N of y in W_β that is evenly covered by f_i. Since N meets $f_i(C)$, some component D of $f^{-1}(N)$ is a subset of C. Therefore $y \in N = f_i(D) \subset f_i(C)$, an impossibility because y is in the boundary of the open set $f_i(C)$. We have shown that for arbitrary i and β, f_i maps each component of $f_n^{-1}(W_\beta)$ homeomorphically onto W_β; i.e., $\{W_\beta\}$ is evenly covered by each f_i.

THEOREM. *Let X be a compact connected ANR and let $f_i\colon X \to X$ be an open locally ε-expansive map for $i = 0, 1, 2, \dots$ such that the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. Then for each fixed point a_0 of f_0, there exist fixed points a_i of f_i such that $\{a_i\}_{i=1}^{\infty}$ converges to a_0.*

PROOF. According to Lemma 2, there exists an open cover $\{W_\beta\}$ of X evenly covered by f_i for $i = 0, 1, 2, \dots$. We may assume that each W_β is evenly covered by π and by each $g_i = f_i\pi$. By Lemma 1, there exist lifts $\tilde{g}_i$: $\tilde{X} \to \tilde{X}$ such that $\{\tilde{g}_i\}_{i=1}^{\infty}$ converges pointwise to $\tilde{g}_0$. We let $\tilde{a}_0$ in $\pi^{-1}(a_0)$ denote the fixed point of $\tilde{g}_0$. Since each $\tilde{g}_i$ is a covering projection, each $\tilde{g}_i$ is a homeomorphism, [5, Corollary 7, p. 77]. Next, we verify that there is a $\delta > 0$ such that $\tilde{g}_i^{-1}$ is locally δ-contractive if i is large enough. From this it would follow that $\{\tilde{g}_i^{-1}\}_{i=1}^{\infty}$ converges pointwise to $\tilde{g}_0^{-1}$.

For each x in X, x is in some W_β. Choose a connected open neighborhood U'_x of x in W_β such that each component of $f_0^{-1}(U'_x)$ has diameter $< \varepsilon$ and each component of $\pi^{-1}(U'_x)$ and $(f_0\pi)^{-1}(U'_x)$ has diameter $< \eta$. Let U_x be a connected open neighborhood of x in U'_x such that its closure, $\overline{U}_x$, is contained in U'_x. Finitely many U_x cover X; call them $\{U_\alpha\}$. Let $\varepsilon' = \min_\alpha\{d(\overline{U}_\alpha, X - U'_\alpha)\}$. Since $f_i \to f_0$ uniformly, there exists an N such that for every $i > N$ and for every $x \in X$, $d(f_i(x), f_0(x)) < \varepsilon'$. We show that for every $i > N$, each component of $f_i^{-1}(U_\alpha)$ has diameter $< \varepsilon$. If $x \in f_i^{-1}(U_\alpha)$, then $f_i(x) \in U_\alpha$ and since $d(f_i(x), f_0(x)) < \varepsilon'$, $f_0(x) \in U'_\alpha$. That is, $x \in f_0^{-1}(U'_\alpha)$. Therefore $f_i^{-1}(U_\alpha) \subset f_0^{-1}(U'_\alpha)$, whose components each have diameter $< \varepsilon$. Similarly, we can show that for every $i > N$, each component of $(f_i\pi)^{-1}(U_\alpha)$ has diameter $< \eta$. Let $\delta < \eta$ be a Lebesgue number for $\{U_\alpha\}$. It follows that for every $i > N$, $\tilde{g}_i^{-1}$ is locally δ-contractive. For, if $\tilde{d}(a, b) < \delta$, then both $\pi(a)$ and $\pi(b)$ belong to the same U_α, and

$$\begin{aligned}\tilde{d}(a, b) &= \tilde{d}\left(\tilde{g}_i\tilde{g}_i^{-1}(a), \tilde{g}_i\tilde{g}_i^{-1}(b)\right)\\ &= d\left(\pi\tilde{g}_i\tilde{g}_i^{-1}(a), \pi\tilde{g}_i\tilde{g}_i^{-1}(b)\right) = d\left(f_i\pi\tilde{g}_i^{-1}(a), f_i\pi\tilde{g}_i^{-1}(b)\right)\\ &> d\left(\pi\tilde{g}_i^{-1}(a), \pi\tilde{g}_i^{-1}(b)\right) = \tilde{d}\left(\tilde{g}_i^{-1}(a), \tilde{g}_i^{-1}(b)\right).\end{aligned}$$

Choose $r > 0$ so small that $r < \delta$ and $K' = \{x \in \tilde{X}: \tilde{d}(\tilde{a}_0, x) \leqslant r\}$ is compact. Since $\tilde{g}_i \to \tilde{g}_0$ uniformly on K, there exists an M such that for all $n > M$ and for $x \in K'$,

$$\delta > r > \tilde{d}\left(\tilde{g}_0(x), \tilde{g}_n(x)\right) > \tilde{d}\left(\tilde{g}_n^{-1}\tilde{g}_0(x), x\right) = \tilde{d}\left(\tilde{g}_n^{-1}\tilde{g}_0(x), \tilde{g}_0^{-1}\tilde{g}_0(x)\right).$$

That is, $\tilde{g}_i^{-1} \to \tilde{g}_0^{-1}$ uniformly on $\tilde{g}_0(K')$. Choose $s > 0$ so that $K = \{x \in \tilde{X}: \tilde{d}(\tilde{a}_0, x) \leqslant s\} \subset \tilde{g}_0(K')$. According to the proof of Theorem 1 in [3], for i large enough, $\tilde{g}_i^{-1}$ maps K into itself and has its unique fixed point $\tilde{a}_i$ in K. This shows $\{\tilde{a}_i\}_{i=1}^{\infty}$ converges to $\tilde{a}_0$. Let $a_i = \pi(\tilde{a}_i)$. It follows that a_i is a fixed point of f_i, and $\{a_i\}_{i=1}^{\infty}$ converges to a_0.

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