A NOTE ON SEMITOPOLOGICAL PROPERTIES

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Abstract. The strongly Hausdorff and Urysohn properties of a topological space are shown to be semitopological properties.

I. Introduction. Levine [6] defined a set \( A \) to be \textit{semiopen} in a topological space if and only if there is an open set \( U \) so that \( U \subseteq A \subseteq \overline{U} \) where \( \overline{\phantom{X}} \) denotes the closure in the topological space.

In [2], \textit{semiclosed} sets, \textit{semi-interior}, and \textit{semiclosure} were defined in a manner analogous to the corresponding concepts of closed, interior, and closure. Then in [3] a property of topological spaces was defined to be a \textit{semitopological property} if it was preserved by \textit{semihomeomorphisms} (bijections so that the images of semipen sets are semiopen and inverses of semiopen sets are semiopen). In [3] the first category, Hausdorff, separable, and connected properties of topological spaces were shown to be semitopological properties.

The new separation axioms (\textit{semi-}\( T_0 \), \textit{semi-}\( T_1 \), and \textit{semi-}\( T_2 \)) defined by Maheshwari and Prasad [7] are also semitopological properties, and Hamlett showed [5] that the property of a topological space being a Baire space is semitopological. In this note two additional separation axioms closely related to the Hausdorff separation axiom are shown to be semitopological properties.

The method of proof, in [3], used to show that the Hausdorff property and connectedness were semitopological properties hinged on the fact that if \([\tau]\) is the equivalence class of topologies on \( X \) which yield the same semiopen sets then there is a finest element of \([\tau]\), denoted by \( F(\tau) \). Also, if \( f: (X, \tau) \to (Y, \sigma) \) is a semihomeomorphism, then \( f: (X, F(\tau)) \to (Y, F(\sigma)) \) is a homeomorphism. A new characterization of \( F(\tau) \) as \( (0 - N|0 \in \tau \text{ and } N \text{ is nowhere dense in } (X, \tau)) \) was given in [1], and this characterization has simplified the proofs given in this paper and it could be used to simplify the proof given in [3] that the Hausdorff separation axiom is a semitopological property.

II. The strongly Hausdorff and Urysohn properties of topological spaces are semitopological properties. The following lemma gives a key part of the proof that the Hausdorff property is a semitopological property in a manner

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Lemma 1. If \( F(\tau) \) is a Hausdorff topology then \( \tau \) is a Hausdorff topology.

Proof. The contrapositive will be proved. If \( \tau \) is not a Hausdorff topology on \( X \), then there are distinct points \( x \) and \( y \) in \( X \) so that for each pair of open sets \( U \in \tau \) and \( V \in \tau \) so that \( x \in U \) and \( y \in V \) it must be the case that \( U \cap V \neq \emptyset \). Now if \( N_1 \subset U \), and \( N_2 \subset V \) are nowhere dense subsets in \( (X, \tau) \) so that \( x \notin N_1 \) and \( y \notin N_2 \), then \( x \in U - N_1 \) and \( y \in V - N_2 \) and we have \( (U - N_1) \cap (V - N_2) = (U \cap V) - (N_1 \cup N_2) \). Furthermore, since \( U \cap V \) is a nonvoid open set and \( N_1 \cup N_2 \) is nowhere dense \( U \cap V \subset N_1 \cup N_2 \) so that \( (U \cap V) - (N_1 \cup N_2) \neq \emptyset \).

Since all open sets in \( F(\tau) \) which contain \( x \) are of the form \( U - N \) where \( U \in \tau \), \( N \subset U \), \( N \) is a nowhere dense set in \( (X, \tau) \) and \( x \notin N \), we see that \( F(\tau) \) is not a Hausdorff topology when \( \tau \) is not.

Hajnal and Juhasz have defined [4] a Hausdorff topological space to be strongly Hausdorff if and only if for each infinite subset \( A \subset X \) there is a sequence \( \{ U_n \mid n \in P \} \) (\( P \) is the set of positive integers) of pairwise disjoint open sets such that \( A \cap U_n \neq \emptyset \) for each \( n \in P \).

Theorem 1. If \((X, \sigma)\) is a strongly Hausdorff space and \( \sigma \in [\tau] \) then \((X, \tau)\) is strongly Hausdorff.

Proof. Since \((X, \sigma)\) is strongly Hausdorff and \( \sigma \subset F(\tau) \), \((X, F(\tau))\) is strongly Hausdorff [4]. Since \((X, F(\tau))\) is Hausdorff, \((X, \tau)\) is Hausdorff by Lemma 1. Now let \( A \subset X \) be any infinite subset of \( X \). Since \((X, F(\tau))\) is strongly Hausdorff there is a sequence of pairwise disjoint open sets \( \{ U_n \mid n \in P \} \) of elements of \( F(\tau) \) such that \( A \cup U_n \neq \emptyset \) for each \( n \in P \). Now for each \( n \in P \) there exists a set \( V_n \in \tau \) and \( N_n \) a nowhere dense set in \((X, \tau)\) so that \( N_n \subset V_n \) and \( V_n - N_n = U_n \) so that \( V_n = U_n \cup N_n \). If \( i \in P \) and \( j \in P \) and \( i \neq j \), \( U_i \cap U_j = \emptyset \), thus we have

\[
V_i \cap V_j = (U_i \cup N_i) \cap (U_j \cup N_j) = (N_i \cap U_j) \cup (U_i \cap N_j) \cup (N_i \cap N_j) \subset N_i \cup N_j.
\]

But \( V_i \cap V_j \) is open in \((X, \tau)\) and \( N_i \cup N_j \) is nowhere dense in \((X, \tau)\) so that \( V_i \cap V_j = \emptyset \). Thus \( \{ V_i \mid i \in P \} \) is a sequence of mutually disjoint elements \( \tau \). Furthermore, we have

\[
A \cap V_n \supset A \cap U_n \neq \emptyset \quad \text{for each } n \in P.
\]

Consequently \((X, \tau)\) is strongly Hausdorff.

Corollary 1. The property of being strongly Hausdorff is a semitopological property.

Proof. If \( f: (X, \tau) \to (Y, \sigma) \) is a semihomeomorphism and \((X, \tau)\) is strongly Hausdorff, then by Theorem 1, \((X, F(\tau))\) is strongly Hausdorff. Since \( f: \)
(X, F(σ)) → (Y, F(σ)) is a homeomorphism [3], (Y, F(σ)) is strongly Hausdorff. Finally by Theorem 1, (Y, σ) is strongly Hausdorff.

A topological space (X, τ) is a Urysohn space if and only if, for each pair of points x ∈ X and y ∈ X there exist open sets U and V so that x ∈ U, y ∈ V, and c(U) ∩ c(V) = Ø.

**Theorem 2.** If (X, σ) is a Urysohn space and σ ∈ [τ] then (X, τ) is Urysohn.

**Proof.** If (X, σ) is Urysohn, then since σ ⊂ F(τ) it follows that (X, F(τ)) is a Urysohn space. Thus the proof will be complete if it can be shown that whenever the finest topology in [τ] is Urysohn then (X, τ) is Urysohn. Since (X, F(τ)) is Urysohn, (X, F(τ)) is Hausdorff so that (X, τ) is Hausdorff by Lemma 1.

If (X, τ) is not a Urysohn space then there exist distinct points a ∈ X and b ∈ X so that for no pair of sets U ∈ τ and V ∈ τ do we have a ∈ U, b ∈ V, c(U) ∩ c(V) = Ø. Now if S ∈ τ and T ∈ τ so that a ∈ S and b ∈ T and S ∩ T = Ø we must still have c(S) ∩ c(T) ≠ Ø. If N₁ and N₂ are nowhere dense in (X, τ) so that a ∉ N₁ and b ∉ N₂ then (S - N₁) ∈ F(τ), (T - N₂) ∈ F(τ), a ∈ (S - N₁) and b ∈ (T - N₂), but

\[ c^*(S - N₁) ∩ c^*(T - N₂) ⊂ c^*(S - N₁) \cap c^*(T - N₂) \]

where c( ) denotes closure in (X, τ) and c*( ) denotes the closure in (X, F(τ)).

Now if q ∈ c(S) ∩ c(T), let W ∈ F(σ) so that q ∈ W. There is a set N₃, disjoint from W, and nowhere dense in (X, τ) so that W ∪ N₃ ∈ τ. Since q ∈ c(S), we have S ∩ (W ∪ N₃) ≠ Ø and S ∩ (W ∪ N₃) ∈ σ. We have

\[ (S - N₁) ∩ (W) = (S ∩ (W ∪ N₃)) - (N₁ ∪ N₃) \]

Notice that since N₁ ∪ N₃ is nowhere dense in (X, τ) and (S ∩ (W ∪ N₃)) ∈ τ, (S ∩ (W ∪ N₃)) - (N₁ ∪ N₃) is not empty. Thus q ∈ c*(S - N₁). By a similar argument q ∈ c*(T - N₂). Consequently, we have c*(S - N₁) ∩ c*(T - N₂) = c(S) ∩ c(T).

Thus, we see that if there are open sets U and V in F(τ) so that a ∈ U, b ∈ V and c*(U) ∩ c*(V) = Ø they cannot be obtained by taking disjoint neighborhoods of a and b in (X, τ) and subtracting nowhere dense sets. On the other hand, the proof of Lemma 1 shows that disjoint elements of F(τ) cannot be obtained from nondisjoint elements of τ by subtracting nowhere dense sets.

Since all open sets in F(τ) are of the form 0 - N where 0 ∈ τ and N is nowhere dense in (X, τ) [1], we see that if (X, τ) is not Urysohn, neither is (X, F(τ)), and the proof of Theorem 2 is complete.

The proof of the following corollary is essentially the same as that of Corollary 1.

**Corollary 2.** The property of being a Urysohn space is a semitopological property.
REFERENCES


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