PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 2, November 1978

REDUCTIONS OF n-FOLD COVERS[1]

SAUL STAHL

ABSTRACT. Motivated by L. Lovász's recent proof of the Kneser conjecture [3], this paper offers another result which relates topological and graph theoretical concepts. A method for converting n-fold covers to $(n-1)$-fold covers is presented. This yields a strengthening of the classical Borsuk, Lusternik and Schnirelmann theorem on covers of spheres. The same conversion also has applications to multicolorings of graphs.

Let $\mathcal{S} = \{S_i\}_{i=1}^{m}$ be a family of subsets of the set S. We say that $\mathcal{S}$ is an *n-fold cover* of S if every element of S is contained in at least n of the S_i. The letters m and n always denote positive integers.

A classical theorem of Borsuk [1] and of Lusternik and Schnirelmann [4] states that in any cover of the k-dimensional sphere by means of $k+1$ closed subsets, at least one of the subsets contains a pair of antipodal points. It is well known that covers employing $k+2$ subsets exist none of whose members contain antipodal points. Examples have been constructed by Borsuk [1] and Gale [2]. The author is indebted to the referee for suggesting the following easily described example. Inscribe a regular $(k+1)$-dimensional simplex inside S^k in $\mathbf{R}^{k+1}$. Surround each of the $k+2$ vertices by a k-dimensional disk in S^k, centered at the vertex and large enough so that it is not quite a hemisphere. These $k+2$ closed disks are obviously a cover of S^k, but none of them contains an antipodal pair of points. However, Corollary 1 to Theorem 1 shows that the classical theorem can be strengthened to an "antipodal points result" for covers of S^k by more than $k+1$ subsets. The basic idea is to add a requirement on the number of times each point is covered.

THEOREM 1. *Let $\mathcal{S} = \{S_i\}_{i=1}^{m}$ be an n-fold cover of the set S, with $m > n$. For each $i \in \{1, 2, \ldots, m-2\}$ define*

$$T_i = \left[\bigcap_{j=1}^{i+1} S_j\right] \cup \left[S_{i+2} \cap \left(\bigcup_{j=1}^{i+1} S_j\right)\right].$$

Then the family $\mathcal{T} = \{T_i\}_{i=1}^{m-2}$ has the following properties:

(a) *$\mathcal{T}$ is an $(n-1)$-fold cover of S*;

(b) *if any two elements of S are both contained in some member of $\mathcal{T}$ then*

Received by the editors October 27, 1977 and, in revised form, February 27, 1978.

AMS (MOS) subject classifications (1970). Primary 05C10.

[1]The author would like to thank Dr. Fred Galvin for many helpful conversations in this context.

they are also both contained in some member of $\mathfrak{S}$.

PROOF. Let x be any element of S. Since $\mathfrak{S}$ is an n-fold cover of S, there exist integers $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant m$ such that $x \in \bigcap_{j=1}^{n} S_{i_j}$. If $i_1 > 1$, then clearly $x \in S_i \cap (\bigcup_{j=1}^{i-1} S_j)$ for all $i \in \{i_2, \ldots, i_n\}$ and consequently x is contained in all the $n - 1$ sets T_i where $i \in \{i_2 - 2, \ldots, i_n - 2\}$. If $i_1 = 1$, let k be the largest integer such that $i_k = k$. Note that $i_j = j$ for $j \in \{1, 2, \ldots, k\}$ and that $i_j \geqslant j + 1$ for $j \geqslant k + 1$. Since $x \in \bigcap_{j=1}^{i} S_j$ for $i \in \{1, 2, \ldots, k\}$ it follows that $x \in T_i$ for $i \in \{1, 2, \ldots, k - 1\}$. Observe that as $m > n$ we may conclude that $m - 2 \geqslant n - 1 \geqslant k - 1$ and so we do indeed have here $k - 1$ members of $\mathfrak{T}$ which contain x. For $j \geqslant k + 1$ we know that $i_j > i_1$ and hence $x \in \bigcup_{i=1}^{i_j-1} S_i$. Since $x \in S_{i_j}$ we conclude again that $x \in T_i$ for $i \in \{i_{k+1} - 2, \ldots, i_n - 2\}$. From the definition of k it follows that the sets $\{1, 2, \ldots, k - 1\}$ and $\{i_{k+1} - 2, \ldots, i_n - 2\}$ are disjoint and so x is again contained in $n - 1$ of the T_i.

(b) Suppose x and y are both in T_i for some $i \in \{1, 2, \ldots, m - 2\}$. If x and y are both in $\bigcap_{j=1}^{i+1} S_j$ or both in $S_{i+2} \cap (\bigcup_{j=1}^{i+1} S_j)$, then x and y are both in S_1 or both in S_{i+2}, respectively. Hence there only remains to be considered the case where $x \in \bigcap_{j=1}^{i+1} S_j$ and $y \in S_{i+2} \cap (\bigcup_{j=1}^{i+1} S_j)$. But then x and y are both in the same S_i for some $i \in \{1, 2, \ldots, i + 1\}$. Thus there always does exist an S_i containing both x and y. Q.E.D.

It should be noted here that the hypothesis $m > n$ only excludes the trivial cover wherein each member of $\mathfrak{S}$ actually equals S. In that case the conclusion of the theorem does not hold, but the situation is clear and uninteresting.

COROLLARY 1. *Given two integers $n \geqslant 1$ and $k \geqslant 0$, let $\mathfrak{S} = \{S_i\}_{i=1}^{2n+k-1}$ be an n-fold cover of the k-dimensional sphere S^k. If each S_i is a closed subset, then some member of $\mathfrak{S}$ contains an antipodal pair of points of S^k.*

PROOF. By induction on n. If $n = 1$, then this is the above mentioned theorem of Borsuk, Lusternik and Schnirelmann. Now assume the validity of this corollary for every $(n - 1)$-fold cover of S^k (where k is fixed) with closed subsets, and let $\mathfrak{S}$ be an n-fold such cover. If $2n + k - 1 \leqslant n$, then necessarily $n = 1$, $k = 0$, and the corollary follows trivially. Else, Theorem 1 applies. Let $\mathfrak{T}$ be the $(n - 1)$-fold cover constructed from $\mathfrak{S}$. It is clear from the definition of the T_i that they are closed subsets, and since $\mathfrak{T}$ contains $2n + k - 1 - 2 = 2(n - 1) + k - 1$ members, it follows from the induction hypothesis that some T_i contains an antipodal pair of points. Part (b) of Theorem 1 implies that this antipodal pair is also contained in some S_j. Q.E.D.

D. Gale [2] has constructed for n, k as above, an n-fold cover of S^k which consists of $2n + k$ open hemispheres. Since these can be easily trimmed down to closed subsets without ceasing to be an n-fold cover, it may be said that Corollary 1 is a best possible result.

Theorem 1 can also be used to give an alternate proof of a graph theoretical theorem first proved in Stahl [5]. An *independent* subset of the graph G is a set of vertices no two of which are adjacent. The *nth chromatic number* of G, $\chi_n(G)$, is the least number of sets in any n-fold covering of G with independent subsets.

COROLLARY 2. *If the graph G contains an edge, then*

$$\chi_n(G) \geqslant 2 + \chi_{n-1}(G) \quad \textit{for } n = 2, 3, 4, \ldots.$$

PROOF. Let $\mathfrak{S} = \{S_i\}_{i=1}^m$ be an n-fold cover of G with independent subsets, where $m = \chi_n(G)$. Since G does contain an edge, none of the S_i equals G, and hence $m > n$. Again Theorem 1 applies and we let $\mathfrak{T}$ be the resulting $(n - 1)$-fold cover of G. If some T_i were not independent it would contain a pair of adjacent vertices of G. Because of property (b), however, such a pair would necessarily be contained in some S_j as well, thus contradicting the independence of the members of $\mathfrak{S}$. Consequently each member of $\mathfrak{T}$ is independent and hence $\chi_{n-1}(G) \leqslant m - 2 = \chi_n(G) - 2$. Q.E.D.

BIBLIOGRAPHY

1. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.

2. D. Gale, *Neighboring vertices on a convex polyhedron*, Linear inequalities and related systems, Ann. of Math. Studies, no. 38, Princeton Univ. Press, Princeton, N.J., 1956, pp. 255–263.

3. L. Lovász, *Kneser's conjecture, homotopy and Borsuk's theorem*, J. Combinatorial Theory Ser. B (to appear).

4. L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Gosudarstvennoe Izdat., Moscow, 1930; rev. French transl., Actualités Sci. Indust., no. 118, Hermann, Paris, 1934.

5. S. Stahl, *n-tuple colorings and associated graphs*, J. Combinatorial Theory Ser. B **20** (1976), 185–203.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045