SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON ČECH'S THEOREM

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In this note we give an unusual proof of a theorem of Čech [1]. Given a completely regular space $Y$, let $\beta Y$ be the Čech-Stone compactification of $Y$.

**Theorem.** If $X$ is a metrizable space which is a $G_b$ in $\beta X$, then $X$ is completely metrizable.

**Proof.** For every space $Y$ we let $C^*(Y)$ be the space of continuous bounded real-valued functions on $Y$, with the usual supremum norm. Note that $\beta f \in C^*(\beta Y)$ if $f \in C^*(Y)$. We will show that $X$ is completely metrizable by finding an embedding $\phi: X \to C^*(\beta X)$ such that $\phi[X]$ is closed in $C^*(\beta X)$. We may assume that $X$ is not compact. Let $\rho$ be a compatible bounded metric on $X$. For $s \in X$ define $\rho_s \in C^*(X)$ by $\rho_s(x) = \rho(s, x)$. Note that $\|\rho_s - \rho_t\| = \rho(s, t)$, for $s, t \in X$. Write $\beta X - X$ as $\bigcup_{n \in \mathbb{N}} F_n$, with each $F_n$ compact and nonempty. For $n \in \mathbb{N}$ define $\alpha_n: X \to \mathbb{R}$ by $\alpha_n(s) = \min\{\beta \rho_s(x): x \in F_n\}$. Then $\alpha_n(s) > 0$ for $s \in X$ (for there is an $\varepsilon > 0$ such that $\{x \in X: \rho(s, x) < \varepsilon\} \cap F_n = \emptyset$), and each $\alpha_n$ is continuous (for one easily checks that $|\alpha_n(s) - \alpha_n(t)| < \|\rho_s - \rho_t\|$ whenever $s, t \in X$). It follows that we can define $\phi_n \in C^*(\beta X)$ by

$$\phi_n(s)(x) = \min\{1, \beta \rho_s(x)/\alpha_n(s)\}.$$

Note that $\phi_n(s)(x) = 1$ if $x \in F_n$, for $s \in X$ and $n \in \mathbb{N}$. We now claim that the function $\phi = \Sigma 2^{-n}(2^{-n} \phi_n: n \in \mathbb{N})$ is the required embedding. Evidently $\phi$ is continuous. If $s, t \in X$ then

$$\|\phi(s) - \phi(t)\| \geq |\phi(s)(t) - \phi(t)(t)|$$

$$= \phi(s)(t) \geq 2^{-1}\min\{1, \rho(s, t)/\alpha_1(s)\}.$$ 

It follows that $\phi$ is one-to-one and that $\phi^{-1}$ is continuous at $\phi(s)$ for each $s \in X$. It remains to show that $\phi[X]$ is closed. Let $f \in \phi[X]^-$ be arbitrary. There is a sequence $\langle x_k \rangle_k$ in $X$ such that $\langle \phi(x_k) \rangle_k$ converges to $f$. Let $x$ be a cluster point of $\langle x_k \rangle_k$ in $\beta X$. Then $f(x) = 0$ since $\phi(x_k)(x_k) = 0$ for each $k \in \mathbb{N}$. Therefore $x \in X$. (For otherwise there is an $n \in \mathbb{N}$ with $x \in F_n$, and...
then \( f(x) > 2^{-n} \) since \( \phi(x_k)(x) > 2^{-n} \phi(x_k)(x) = 2^{-n} \) for all \( k \in N \).

Consequently \( f = \phi(x) \in \phi[X] \). □

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REFERENCES


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