THE LOCAL HILBERT FUNCTION OF A PAIR OF PLANE CURVES

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Abstract. Let $R$ be the local ring of a pair of plane curves at a point. In this paper it is proved that the Hilbert function of such a ring changes by at most one at each stage, and it is essentially nonincreasing.

1. Introduction. Let $\mathfrak{O}$ be an equicharacteristic two dimensional regular local ring and $f$ and $g$ be two elements of $\mathfrak{O}$ which generate an ideal $\mathfrak{U}$ primary for the maximal ideal of $\mathfrak{O}$. Let $R$, $\text{Gr}(\mathfrak{O})$ and $\text{Gr}(R)$ denote respectively the residue-class ring $\mathfrak{O}/\mathfrak{U}$ and the graded rings of $\mathfrak{O}$ and $R$. $\text{Gr}^i(R)$ denotes the $i$th homogeneous component of $\text{Gr}(R)$ and $H_i(R)$ denotes its vectorspace dimension over $\text{Gr}^0(R)$. The function $H_i(R)$ is referred to as the Hilbert function of $R$. The orders of $f$ and $g$ are denoted by integers $n$ and $m$ respectively.

The following questions were raised by Professor Abhyankar.

Question 1. Is $\sum_{i=0}^{n+m-1} H_i(R) > mn$ when $f$ and $g$ are tangential, i.e., when their initial forms have a common factor?

Question 2. Is $|H_{i+1}(R) - H_i(R)| < 1$ for all nonnegative integers $i$?

These questions are answered in the affirmative and the following theorem is proved.

Theorem. $0 < H_i(R) - H_{i+1}(R) < 1$ for $i > \min(n, m)$.

We will denote by $x, y$ a regular system of parameters in $\mathfrak{O}$ and the residue field of $\mathfrak{O}$ by $k$. $\text{Gr}(\mathfrak{O})$ will be identified with the polynomial ring $k[X, Y]$ with $X$ and $Y$ denoting initial forms of $x, y$ respectively. With this identification the initial ideal $\mathfrak{U}$ of $\mathfrak{U}$ will be a homogeneous ideal in $k[X, Y]$. The $i$th homogeneous component of $\mathfrak{U}$ and its dimension over $k$ are denoted by $\mathfrak{U}_i$ and $\dim(\mathfrak{U}_i)$ respectively. $F$ and $G$ denote initial forms of $f$ and $g$. The definitions of the terms used here can be found in [3].

2. For the purpose of the following lemma $\mathfrak{O}$ will denote the power series ring $k[[x, y]]$ and the field $k$ is assumed to be infinite.

Lemma. Let $f$ and $g$ be such that $m > n$, $g = xg_1$ with $g_1$ in $\mathfrak{O}$ and $X$ does not divide $F$. Let $\mathfrak{U}$ be the ideal generated by $f$ and $g_1$ and $\mathfrak{U}$ be its initial ideal. Then $\dim(\mathfrak{U}_{i+1}) = \dim(\mathfrak{U}_i) + 1$ for all $i > n$. 

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Proof. If \( W_1, W_2, \ldots, W_r \) is a basis of \( \mathfrak{U}_i \) then \( XW_1, XW_2, \ldots, XW_r \) and \( Y^{i+1-n}F \) are elements in \( \mathfrak{U}_{i+1} \) which are linearly independent. Hence it readily follows that \( \dim(\mathfrak{U}_{i+1}) \geq \dim(\mathfrak{U}_i) + 1 \).

Let \( t_1 = Y^{i+1-n}F \). We prove that there exist elements \( t_2, t_3, \ldots, t_s \) in \( \mathfrak{U} \), each divisible by \( x \) and such that their initial forms together with \( Y^{i+1-n}F \) form a basis of \( \mathfrak{U}_{i+1} \). First choose elements \( t^*_2, t^*_3, \ldots, t^*_s \) such that their initial forms and \( Y^{i+1-n}F \) form a basis of \( \mathfrak{U}_{i+1} \). If \( t^*_j \) is divisible by \( x \) then take \( t_j = t^*_j \), otherwise \( t_j \) is chosen as follows. By Weierstrass preparation theorem,

\[
t_1 = \delta[y^{i+1} + c_1(x)y^i + \cdots + c_{i+1}(x)]
\]

and

\[
t^*_j = \eta[y^p + d_1(x)y^{p-1} + \cdots + d_p(x)],
\]

where \( \delta \) and \( \eta \) are units and \( c_1(x), \ldots, c_{i+1}(x), d_1(x), \ldots, d_p(x) \) are power series in \( x \) divisible by \( x \). Since \( m \geq n, p \) will be bigger than or equal to \( i + 1 \). Let \( t_j = t^*_j - \eta \delta^{-1} y^{p-i-1}t_1 \). The elements \( t_2, t_3, \ldots, t_s \) chosen this way are clearly divisible by \( x \) and their initial forms together with \( Y^{i+1-n}F \) form a basis of \( \mathfrak{U}_{i+1} \).

Let \( t_j = af + bg \) for \( j \geq 2 \). For \( j \geq 2, t_j \) is divisible by \( x \), write \( t_j = xw_j \) for \( j \geq 2 \), also \( g = xg_1 \). We get \( xw_j = af + xb_jg \), which implies that \( x \) divides \( a_j \) and consequently \( w_j \) belongs to \( \mathfrak{U} \). The initial forms of \( w_j \) are in \( \mathfrak{U}_i \) and linearly independent and hence \( \dim(\mathfrak{U}_i) + 1 \geq \dim(\mathfrak{U}_{i+1}) \). This completes the proof of the lemma that \( \dim(\mathfrak{U}_{i+1}) = \dim(\mathfrak{U}_i) + 1 \) for \( i \geq n \).

3. Definition. The initial forms of a pair of generators of \( \mathfrak{U} \) are said to be irredundant if none of them is a multiple of the other.

Observe that if a pair of generators of \( \mathfrak{U} \) is such that their initial forms are irredundant then the sum of the degrees of the initial forms and the degree of the g.c.d. of the initial forms are two well-defined numbers dependent only on the ideal \( \mathfrak{U} \). These numbers will be denoted by \( \alpha(\mathfrak{U}), \beta(\mathfrak{U}) \) respectively and are used for induction in the proof of the following theorem.

Theorem. \( 0 < H_i(R) - H_{i+1}(R) < 1 \) for all \( i \geq \min(n, m) \).

Proof. The completion \( \hat{R} \) of \( R \) is isomorphic to \( \hat{\mathfrak{U}} / \mathfrak{U} \), where \( \hat{\mathfrak{U}} \) is the completion \( \mathfrak{U} \) and \( \mathfrak{U} = \mathfrak{U} / \mathfrak{U} \) and also since \( R \) and \( \hat{R} \) have isomorphic graded rings \( H_i(R) = H_i(\hat{R}) \). Hence without loss of generality we assume that the rings \( \mathfrak{U} \) and \( R \) are complete. By Cohen’s structure theorem \( \mathfrak{U} \) is then isomorphic to the power series ring \( k[[x, y]] \).

Since \( \text{Gr}(R) \simeq k[X, Y]/\mathfrak{U} \) and \( H_i(R) = i + 1 - \dim(\mathfrak{U}_i) \), the theorem is equivalent to proving \( 1 < \dim(\mathfrak{U}_{i+1}) - \dim(\mathfrak{U}_i) < 2 \) for all \( i > \min(n, m) \). The computation of \( \dim(\mathfrak{U}_i) \) is fairly straightforward for \( i < \min(n, m) + 1 \) and hence the above inequality easily checked for \( i = \min(n, m) \). In view of this it is sufficient to prove \( 1 < \dim(\mathfrak{U}_{i+1}) - \dim(\mathfrak{U}_i) < 2 \) for \( i > \min(n, m) + 1 \). We give a proof of this statement assuming that the field \( k \) is infinite.

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The proof when \( k \) is finite can be reduced to that case by extending \( k \) by adjoining an indeterminate.

Let \( T_1, T_2, \ldots, T_r \) be a basis over \( k \) of \( \mathfrak{U}_i \) for \( i > \min(n, m) + 1 \). If \( T_j \) is an element of the basis least divisible by \( X \) then \( YT_1, XT_1, XT_2, \ldots, XT_r \) are linearly independent. These elements are in \( \mathfrak{U}_{i+1} \) and we have that \( \dim(\mathfrak{U}_{i+1}) - \dim(\mathfrak{U}_i) > 1 \) for \( i > \min(n, m) + 1 \).

It remains to be proved that \( \dim(\mathfrak{U}_{i+1}) - \dim(\mathfrak{U}_i) < 2 \) for \( i > \min(n, m) + 1 \). Without loss of generality we can assume that \( n < m \) and that the initial forms of \( f \) and \( g \) are irredundant. Since the field \( k \) is infinite we can, if necessary by making suitable linear transformation in \( x \) and \( y \), assume that \( X \) does not divide the initial form of \( f \). Then we show that there exists a \( g^* \) divisible by \( x \) and such that \( f \) and \( g^* \) generate \( \mathfrak{U} \) and their initial forms are irredundant. If \( x \) divides \( g \) then there is nothing to prove hence assume that \( x \) does not divide \( g \). By Weierstrass preparation theorem

\[
f = \delta \left[ y^p + c_1(x)y^{p-1} + \cdots + c_n(x) \right] \quad \text{and} \quad g = \eta \left[ y^p + d_1(x)y^{p-1} + \cdots + d_p(x) \right]
\]

where \( p \) is some integer bigger than or equal to \( m \), \( \delta, \eta \) are units and \( c_1(x), \ldots, c_n(x), d_1(x), \ldots, d_p(x) \) are power series in \( x \) divisible by \( x \). Then \( g^* = g - \eta \delta^{-1} y^p - n \) is as desired. Let \( g^* = xg_1 \) and \( \mathfrak{V} \) be the ideal generated by \( f \) and \( g_1 \). In view of the lemma it is enough to prove that \( \dim(\mathfrak{V}_{i+1}) - \dim(\mathfrak{V}_i) < 2 \) for \( i > n + 1 \). Since \( \mathfrak{V} \) is generated by \( f \) and \( g_1 \), \( \beta(\mathfrak{V}) < \beta(\mathfrak{U}) \) and if \( \beta(\mathfrak{V}) = \beta(\mathfrak{U}) \) then \( \alpha(\mathfrak{V}) < \alpha(\mathfrak{U}) \), hence the proof of the theorem follows by induction.

Now in the following corollaries we answer the questions raised by Professor Abhyankar. If \( f \) and \( g \) are nontangential, i.e., if their initial forms are coprime then \( H_i(\mathfrak{R}) \) depends only on the orders of \( f \) and \( g \), and we will denote it by \( H_i(n, m) \). It is easily checked that \( \sum_{i=0}^{\infty} H_i(n, m) = mn \) and \( H_i(n, m) = 0 \) for all \( i > m + n - 1 \).

**Corollary 1.** If \( f \) and \( g \) are tangential then \( \sum_{i=0}^{m+n-1} H_i(R) > mn \).

**Proof.** Let \( p > 0 \) be the degree of the g.c.d. of the initial forms of \( f \) and \( g \). It is easily checked that \( H_i(R) = H_i(n, m) \) for \( 0 < i < m + n - p - 1 \) and \( H_{m+n-p}(R) = H_{m+n-p}(n, m) + 1 \). By the above theorem, \( H_i(R) \) decreases by at most one at each stage for \( i > \min(n, m) \) whereas \( H_i(n, m) \) decreases by one at each stage for \( i > \min(n, m) \) until it becomes zero. It follows that \( H_i(R) > H_i(n, m) \) for all \( i \) and

\[
\sum_{i=0}^{m+n-1} H_i(R) > \sum_{i=0}^{m+n-1} H_i(n, m) = mn.
\]

**Corollary 2.** \( |H_{i+1}(R) - H_i(R)| < 1 \) for all nonnegative integers.

**Proof.** For \( 0 < i < \min(n, m) \) this is checked by direct calculation and for \( i > \min(n, m) \) this follows from the theorem.
We deduce the following result of Bertini cited in [1].

**Corollary 3.** Let $\alpha$ be the intersection multiplicity of $f$ and $g$ and $e$ be the largest integer such that $H_i(R) > 0$ for all $i < e$ then $e \leq \alpha + m + n - mn - 2$.

**Proof.** Since, as observed in the proof of Corollary 1, $H_i(R) > H_i(n, m)$ for all $i$ and $H_i(n, m) = 0$ for $i > m + n - 1$ hence $\sum_{i=0}^{\alpha} H_i(R) > \sum_{i=0}^{\alpha + n - 2} H_i(n, m) + (e - m - n + 2)$. But $\sum_{i=0}^{\alpha} H_i(R) = \alpha$, the intersection multiplicity and $\sum_{i=0}^{\alpha + n - 2} H_i(n, m) = mn$ hence the result follows.

**References**


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