ON FINITE SIMPLE GROUPS WITH A SELF-CENTRALIZATION SYSTEM OF TYPE $(2(n))$

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Abstract. Let $G$ denote a simple group with a self-centralization system of type $(2(n))$, where $n > 3$. Let $X_1$ denote an exceptional character of $G$, then $X_1(1) = kn + 2e$ where $e = \pm 1$. It is known that

$$|G| = nX_1(1)(X_1(1) - e)(kn + 1)$$

where $l$ is a nonnegative integer. In this paper $G$ is classified if $l = 0, e = 1$ and $X_1(1)$ is odd.

Let $G$ be a finite simple group, a proper subgroup $A$ of $G$ is called a CC subgroup if $CG(a) \subseteq A$ for all $a \in A^u$. If $|N_G(A)|/|A| = 2$ and $|A| = n$, then $G$ is said to have a self-centralization system of type $(2(n))$. The classification of simple groups with a self-centralization system of type $(2(n))$ is still incomplete. If $n = 3$, then $G \cong PSL(2, q), q = 5$ or 7 [2]. If $n > 3$, it is well known [5] that $G$ has $(n - 1)/2$ irreducible characters $X_i$, and one nonprincipal irreducible character $Y$ such that $X_1(1) = kn + 2e$, $Y(1) = kn + e$ where $e = \pm 1$ and $|G| = nX_1(1)Y(1)(ln + 1)$ where $l$ is a nonnegative integer. In all the known simple groups of type $(2(n))$, $l = 0$ [5]. In this paper we classify all simple groups $G$ with a self-centralization system of type $(2(n))$ where $l = 0, e = 1$, and $X_1(1)$ is odd. In particular we prove the following:

**Theorem A.** Let $G$ be a finite simple group with a self-centralization system of type $(2(n))$ where $n > 5$. Let $A$ be a subgroup of order $n$ and let $X_1$ be an exceptional character of $G$ associated with $N(A)$. Assume $X_1(1) = kn + 2$, $X_1(1)$ is odd and $|G| = n(kn + 2)(kn + 1)$, then $G$ is isomorphic to $Sz(q)$ or $PSL(2, 2^r)$.

Let $G \in$ Hypothesis A if $G$ satisfies the hypothesis of Theorem A but not the conclusion. Let $\tau$ be an involution in $N(A)$, Theorem 5.1 [5] implies $G$ has one class of involutions. If $S$ is a set, let $|S|$ denote the number of elements in $S$.

Assume $G \in$ Hypothesis A.

If $|C_G(\tau)| = 2^r$, then [6] implies either $G$ satisfies Theorem A, $G \cong PSL(3, 4)$ or $G \cong PSL(2, q)$ where $q$ is odd, $n = (q + 1)/2$ or $(q - 1)/2$ where $n$ is odd. However $X_1(1)$ is even for $PSL(3, 4)$ or $PSL(2, q), q$ odd. Let $G_2$ be a Sylow 2 subgroup of $G$; if $G_2$ is abelian, then [7] again implies a
contradiction. Hence, \( G = \) Hypothesis A implies \(|C_G(\tau)| \neq |G_2|\) and \(G_2\) is not abelian.

If \( z \in C_G(\tau)^g \), let \( F_z = \{ \tau^\sigma | \sigma \in C_G(z) \} \).

**Lemma 1.** Assume \( G = \) Hypothesis A; then there is an element \( x \in C_G(\tau)^g \) such that \( x \) has odd order and \( F_x \neq F_\tau \).

**Proof.** Assume \( x \) an element of odd order in \( C_G(\tau)^g \) implies \( F_x = F_\tau \). \( G = \) Hypothesis A implies \(|F_\tau| \neq 1\). Let \( \tau_2 \in F_\tau \setminus \{ \tau \} \); then \( \tau_2 = \tau^g \) for some \( g \in G \). Now \( \tau_2 \in F_x \) implies \( x \in C_G(\tau_2) = (C_G(\tau))^g \). Hence \( x = y^g \) where \( y \) has odd order and \( y \in C_G(\tau)^g \). Therefore \( F_x = F_y^g = (F_y)^g = (F_\tau)^g = F_\tau^g \). Hence \( F_\tau = F_\tau^g \). Let \( \langle F_\tau \rangle \) be the group generated by \( F_\tau \), then \( F_\tau^g = F_\tau \) implies \( \langle F_\tau \rangle \) is an abelian 2-group. Thus \( \Omega_1(G_2) \) is abelian and Goldschmidt [3] implies \( G \notin \) Hypothesis A.

**Proof of Theorem A.** We will assume \( G = \) Hypothesis A and obtain a contradiction. Let \( Y \) be the nonprincipal nonexceptional character associated with \( N(A) \). Since \(|G_2| | Y(1)|, Y(\tau) = 0 \) and Lemma 6 of [4] implies \(|C_G(\tau)| = Y(1)|\). Theorem 17.4 [1] implies \( Y(z) = 0 \) for \( z \in C_G(\tau)^g \). Since \( \epsilon = 1 \), Lemma 4 [4] implies \( X_i(z) = 1 \) for \( z \in C_G(\tau)^g \), and \( i = 1, \ldots, (n-1)/2 = t \).

Let \( T \) denote the principal character of \( C_G(\tau) \) and let \( T^* \) denote the character of \( G \) induced by \( T \). Let \( 1_G \) denote the principal character of \( G \). Frobenius Reciprocity now implies \((T^*, X_i) = 2 \) for \( i = 1, \ldots, t \), and \((T^*, Y) = (T^*, 1_G) = 1\). Now \( T^*(1) = |G|/Y(1) = nX_1(1) \) and \( \sum_{i=2}^{t} 2X_i(1) + Y(1) + 1_G(1) = nX_1(1) \) imply \( T^* = \sum_{i=2}^{t} 2X_i + Y + 1_G \). Hence, \( z \in C_G(\tau)^g \) implies \( T^*(z) = n \).

If \( z \in C_G(\tau)^g \), then \( T^*(z) \) is the number of involutions in \( C_G(z) \); hence \(|F_\tau| = n \). Let \( x \in C_G(\tau)^g \) where \( x \) has odd order and let \( z = x\tau \). Now \( F_z = F_x \cap F_\tau \). Hence \(|F_z| = n = |F_\tau| = |F_x| \) implies \( F_\tau = F_z = F_x \). This contradicts Lemma 1.

**References**