ON THE LOCAL SPECTRA OF SEMINORMAL OPERATORS

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ABSTRACT. Two theorems on the local spectra of seminormal operators are deduced. In the first theorem it is shown that when \( T \) is hyponormal any solution of the equation \((T - \lambda)x(\lambda) = x\) on an open set in the plane is necessarily analytic. The second theorem establishes the existence of vectors with small local spectra for a cohyponormal operator with a finite rank self-commutator.

Let \( T \) be an operator on a Hilbert space \( \mathcal{H} \). For \( x \) a vector in \( \mathcal{H} \) the local resolvent \((T - \lambda)^{-1}x\) is analytic on the resolvent set \( \rho(T) \). In case any two analytic extensions of \((T - \lambda)^{-1}x\) agree on their common domain, then the operator \( T \) is said to have the single valued extension property. When \( T \) has the single valued extension property the domain of the maximal analytic continuation \( \tilde{x} \) of the local resolvent is denoted by \( \rho_T(x) \). The complementary set \( \sigma_T(x) = \mathcal{C} \setminus \rho_T(x) \) is called the local spectrum of \( x \). If \( T \) has the single valued extension property, then for \( \delta \) a closed subset in the plane \( X_T(\delta) \) will denote the linear manifold consisting of vectors \( x \) such that \( \sigma_T(x) \) is contained in \( \delta \).

An operator \( S \) on \( \mathcal{H} \) is called seminormal in case its selfadjoint self-commutator \( D = S*S - SS* \) is semidefinite. In case \( D > 0 \), the operator \( S \) is called hyponormal and when \( D < 0 \), the operator \( S \) is called cohyponormal. There is a marked difference between the local spectral theories for hyponormal and cohyponormal operators. It is known that if \( H \) is hyponormal, then \( H \) has the single valued extension property. Further \( X_H(\delta) \) is closed for all closed \( \delta \subset \mathcal{C} \). This latter result appears in Stampfli [7] with the hypothesis that the spectrum \( \sigma(H) \) consists of continuous spectrum. Radjabalipour [6] has shown that the latter hypothesis on the spectrum of \( H \) is unnecessary. It is apparently unknown (excepting trivial cases where \( H^* \) has eigenvalues) whether \( \delta \) can be chosen so that \( X_H(\delta) \) provides a nontrivial invariant subspace for the operator \( H \). On the other hand when \( C \) is a nonnormal cohyponormal operator with the single valued extension property there are nonzero vectors with local spectra a proper subset of \( \sigma(C) \). Whether \( X_C(\delta) \) is closed for closed \( \delta \subset \mathcal{C} \) is not known. These vectors with small local spectra are provided by a result of Putnam [5] which asserts that any vector in the range of \( CC^* - C^*C \) belongs to the range of \( C - \lambda \), for all \( \lambda \in \mathcal{C} \). As we shall see below there are no nonzero vectors in the range of \( H - \lambda \), for all

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\( \lambda \in \mathbb{C} \), when \( H \) is a hyponormal operator.

Let \( T \) be an operator on \( \mathcal{K} \) and \( \delta \subset \mathbb{C} \) closed. The notation

\[
Z_T(\delta) = \bigcap_{\lambda \notin \delta} (T - \lambda)\mathcal{K}
\]

will be employed.

**Theorem 1.** Let \( H \) be a hyponormal operator on \( \mathcal{K} \) and \( \delta \subset \mathbb{C} \) closed. Then

\[
Z_H(\delta) = X_H(\delta).
\]

The above result answers a question of Radjabalipour [6]. Theorem 1 improves a result of Stampfli and Wadhwa [8]. The possibility that \( H \) be a normal operator is not ruled out. For the case where \( H \) is normal the result in Theorem 1 appears in Putnam [6] and implicitly in [2].

An immediate corollary is the following.

**Corollary.** Let \( H \) be hyponormal. Then \( Z_H(\phi) = (0) \).

Although in the nonnormal cohyponormal case it is possible to find vectors with local spectra a proper subset of the spectrum, it is not easy to control the location of the local spectra. In this direction we establish:

**Theorem 2.** Let \( C \) be a cohyponormal operator on \( \mathcal{K} \) possessing the single valued extension property and having a finite rank self-commutator. Let \( \Delta \) be an open disc such that \( \Delta \cap \sigma(C) \neq \emptyset \). There exists a vector \( x \) in \( \mathcal{K} \) such that \( \sigma_C(x) \subset \Delta \).

1. **Preliminaries to Theorem 1.** In this section we record lemmas needed for the proof of Theorem 1.

**Lemma 1.** Let \( T \) be an operator without point spectra and \( x \in \mathcal{K} \). Let \( \Delta_x \) be the set of \( \lambda \) in \( \mathbb{C} \) for which there is a vector \( x(\lambda) \) satisfying \( (T - \lambda)x(\lambda) = x \). Then \( ||x(\lambda)|| \) is a lower semicontinuous function on \( \Delta_x \).

**Proof.** Write \( T - \lambda \) as \( T_\lambda \). Let \( T_\lambda \) be the polar factorization of \( T_\lambda \). The operator \( U_\lambda \) is coisometric with null space the same as \( T_\lambda \) and \( P_\lambda = (T_\lambda T_\lambda^*)^{1/2} \). Let \( P_\lambda = \int t \, dE^\lambda(t) \) denote the spectral resolution of \( P_\lambda \). It is easy to verify that if \( (T - \lambda)x(\lambda) = x \), then

\[
||x(\lambda)||^2 = \int t^{-2}d||E^\lambda(t)x||^2.
\]

This last identity and in fact the lower semicontinuity of \( ||x(\lambda)|| \) follow from the equalities

\[
\int t^{-2}d||E^\lambda(t)x||^2 = \lim_{\mu \to 0^+} \int (t + \mu)^{-2}d||E^\lambda(t)x||^2 = \lim_{\mu \to 0^+} ((P_\lambda + \mu)^{-2}x, x).
\]

The lower semicontinuity is implied by the continuity of \( ((P_\lambda + \mu)^{-2}x, x) \) as a function of \( \lambda \), when \( \mu > 0 \) is fixed. This ends the proof.

The following lemma appears in Stampfli [7, Theorem 1]. Actually as stated
in Stampfli one requires the spectrum of $T$ to consist of continuous spectrum. This hypothesis is removed in Radjabalipour [6].

**Lemma 2.** Let $H$ be hyponormal without eigenvalues and let $x \in \mathcal{H}$. Then

$$\|x(\lambda)\| \leq \frac{\|x\|}{\text{dist}(\lambda, \sigma_H(x))}.$$ 

The final lemma which we need is stated in Radjabalipour [6, Theorem 1] (see, also Stampfli and Wadhwa [8]).

**Lemma 3.** Let $H$ be a hyponormal operator without eigenvalues and $\delta \subset \mathbb{C}$ be closed. Suppose there is a bounded function $x(\lambda)$ satisfying $(H - \lambda)x(\lambda) = x$, $\lambda \notin \delta$. Then $x(\lambda)$ is analytic on $\mathbb{C} \setminus \delta$.

**2. Proof of Theorem 1.** In the proof of the theorem it can be assumed that the operator $H$ has no eigenvalues. Let $\Delta$ be a closed disc. Assume that $x \in H$ satisfies $x \in \bigcap_{\lambda \in \Delta}(H - \lambda)\mathcal{H}$. For $\lambda \in \Delta$, the vector $x(\lambda)$ will be assumed to satisfy $(H - \lambda)x(\lambda) = x$. It will be shown that the interior $\Delta^0$ of $\Delta$ is contained in $\rho_H(x)$. The theorem follows easily from this last statement.

Suppose to the contrary that $\Delta^0 \cap \sigma_H(x) \neq \emptyset$.

For $n = 1, 2, \ldots$, set $F_n = \{\lambda \in \Delta \cap \sigma_H(x): \|x(\lambda)\| < n\}$. It follows from the lower semicontinuity of $\|x(\lambda)\|$ (Lemma 1) that $F_n$ is closed and by hypothesis $\bigcup_{n=1}^{\infty} F_n = \Delta \cap \sigma_H(x)$. The Baire Category Theorem implies that for some $m$ the set $F_m \cap \Delta^0$ has interior in the relative topology on $\Delta \cap \sigma_H(x)$. This means there is an open disc $D \subset \Delta^0$ with center in $\sigma_H(x)$ so that $\|x(\lambda)\| < m$, for all $\lambda \in D \cap \sigma_H(x)$. Let $D'$ be the disc with same center as $D$ and radius equal to one-half the radius of $D$. Let $\lambda_0 \in D' \cap \rho_H(x)$ (if such a $\lambda_0$ exists) and choose $\gamma_0$ in $D \cap \sigma_H(x)$ such that $|\lambda_0 - \gamma_0| = \text{dist}(\lambda_0, \sigma_H(x))$. Set $z_0 = x(\gamma_0) = (H - \gamma_0)^{-1}x$. It is easy to see that $\sigma_H(x) = \sigma_H(z_0)$ and for $\lambda \in \rho_H(x)$

$$z(\lambda) = \frac{x(\lambda) - x(\gamma_0)}{\lambda - \gamma_0}$$

satisfies $(H - \lambda)z(\lambda) = z_0$. It follows from Lemma 2 that

$$\|z(\lambda_0)\| = \frac{\|x(\lambda_0) - z_0\|}{|\lambda_0 - \gamma_0|} \leq \frac{\|z_0\|}{\text{dist}(\lambda_0, \sigma_H(z_0))}.$$ 

Since $\text{dist}(\lambda_0, \sigma_H(z_0)) = \text{dist}(\lambda_0, \sigma_H(x)) = |\lambda_0 - \gamma_0|$, we have $\|x(\lambda_0)\| < 2\|z_0\| < 2m$. In any case $\|x(\lambda)\| < 2m$, for all $\lambda \in D'$, and from Lemma 3 we conclude that $x(\lambda)$ is analytic on $D'$. This contradicts the assumption that the center of $D'$ belongs to $\sigma_H(x)$ and ends the proof.

**3. Proof of Theorem 2.** Let $C$ be a cohyponormal operator with the single valued extension property. It will be assumed that the self-commutator of $C$ has finite rank $N$ and we will write $D = CC^* - C^*C = \sum_{i=1}^{N} \langle \cdot, \psi_i \rangle \psi_i$ where $\psi_i$
is orthogonal to $\psi_j$ for $i \neq j$. Let $\pi_0(C)$ denote the collection of eigenvalues of $C$. Putnam [5] has shown that there are weakly continuous functions $\tilde{\psi}_i: C \setminus \pi_0(C) \to \mathbb{C}$ such that $(C - z)\tilde{\psi}_i(z) = \psi_i$, $\|\tilde{\psi}_i(z)\| \leq 1$, $i = 1, \ldots, N$, $z \in C \setminus \pi_0(C)$. Below it will be shown that if $\Delta$ is a disc with $\Delta \cap [\sigma(C) \setminus \pi_0(C)] \neq \emptyset$, then one of the functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_N$ fails to be analytic on $\Delta$.

In the remainder of this section it will be assumed that the operator $C$ is completely nonnormal. This means that there are no nontrivial reducing subspaces of $\mathcal{H}$ on which $C$ is a normal operator. This ensures (among other things) that $\pi_0(C^*) = \emptyset$. Let $z \in \sigma(C) \setminus \pi_0(C)$ and let $C - z = W_z P_z^{1/2}$ be the polar factorization of $C - z$; here, $P_z = (C - z)^*(C - z)$ and $W_z$ is unitary. It is easy to verify that $W_z P_z = (P_z + D)W_z$.

A result of Krein [3] asserts that existence of a measurable function $\delta_z$ with compact support in $[0, \infty)$ satisfying $0 < \delta_z < N$ such that

$$\det\left[I_N - \left[\left((P_z + D - \lambda)^{-1}\psi_i, \psi_j\right)\right]_{N \times N}\right] = \exp\left\{-\int_0^\infty \frac{\delta_z(t)}{t - \lambda} \, dt\right\},$$

(1)

for all $\lambda$ such that $\text{Re} \lambda \notin [0, \infty)$. In the preceding equation $I_N$ denotes the $N \times N$ identity matrix and $\left[\left((P_z + D - \lambda)^{-1}\psi_i, \psi_j\right)\right]_{N \times N}$ is the $N \times N$ matrix with $ij$ entry $((P_z + D - \lambda)^{-1}\psi_i, \psi_j)$.

We would like to take the limit in equation (1) as $\lambda \to 0^-$. Our argument proceeds as in Putnam [5]. One notes for $k = 1, \ldots, N$ and $\lambda < 0$

$$\left\|(P_z + D - \lambda)^{-1/2}\psi_k\right\|^2 = \left\|(P_z + D - \lambda)^{-1}\psi_k, \psi_k\right\|^2$$

$$\leq (D - \lambda)^{-1}\psi_k, \psi_k) = \frac{\|\psi_k\|^2}{\|\psi_k\|^2 - \lambda} < 1.$$

It follows that $\psi_k$ belongs to the domain of $(P_z + D)^{-1/2}$ and

$$\lim_{\lambda \to 0^-} (P_z + D - \lambda)^{-1/2}\psi_k = (P_z + D)^{-1/2}\psi_k, \quad k = 1, \ldots, N.$$

Taking the limit as $\lambda \to 0^-$ in equation (1) one obtains

$$\det\left[I_N - \left[\left((P_z + D)^{-1/2}\psi_i, (P_z + D)^{-1/2}\psi_j\right)\right]_{N \times N}\right] = \exp\left\{-\int_0^\infty \frac{\delta_z(t)}{t} \, dt\right\}.$$

Using the facts that $W_z W_z^* = I$ and $W_z^*(P_z + D)^{-1/2}\psi_k = \tilde{\psi}_k(z)$ we can write

$$\det\left[I_N - \left[\left(\tilde{\psi}_i(z), \tilde{\psi}_j(z)\right)\right]_{N \times N}\right] = \exp\left[-\int_0^\infty \frac{\delta_z(t)}{t} \, dt\right].$$

(2)

Carey and Pincus [1] have identified the function $\delta_z$ in the following manner. There exists an integrable function $G^\delta_z$ on the cylinder $[0, \infty) \times T$ such that
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\[
\det \left[ I_N + \left( W_z (P_z + D - \lambda)^{-1}(W_z - \tau)^{-1}\psi_i, \psi_j \right)_{N \times N} \right] \\
= \exp \frac{1}{2\pi i} \int_0^\infty \int_0^{2\pi} G_z^P (t, e^{i\theta}) \frac{de^{i\theta}}{e^{i\theta} - \tau} \frac{dt}{t - \lambda}
\]

for all \( \lambda \in [0, \infty) \) and \(|\tau| \neq 1\). The identity

\[
\delta_z (t) = \frac{1}{2\pi} \int_0^{2\pi} G_z^P (t, e^{i\theta}) d\theta \quad \text{a.e.} \quad (3)
\]

is established in [1]. The function \( G_z^P \) is referred to as the polar principal function for the operator \( C_z \). There is a second principal function \( G \) defined for the operator \( C \). Write \( C = U - iV \) where \( U, V \) are selfadjoint. There exists [1] an integrable function \( G \) on \( \mathbb{R}^2 \) such that

\[
\det \left[ I_N + \left( (V - v)^{-1}(U - u)^{-1}\psi_i, \psi_j \right)_{N \times N} \right] \\
= \exp \int \int G (\nu, \mu) \frac{d\nu}{\nu - v} \frac{d\mu}{\mu - u} \cdot 
\]

The basic result relating \( G \) and \( G_z^P \) is the identity

\[
G_z^P (t, e^{i\theta}) = G (\nu, \mu) \quad \text{a.e.} \quad (4)
\]

where \( \mu + iv - \bar{z} = \sqrt{t} e^{i\theta} \) [1].

Then

\[
\int_0^\infty \frac{\delta_z (t)}{t} \ dt = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} G_z^P (t, e^{i\theta}) \ dt \ d\theta \\\n= \frac{1}{\pi} \int \int \frac{G (\nu, \mu)}{|\mu + iv - \bar{z}|^2} \ d\mu \ d\nu.
\]

There is a subset \( B \subset A = \{ z = \mu + iv: G (\nu, \mu) \neq 0 \} \) such that the last integral is infinite at every \( \bar{z} \in B \), moreover the planar measure of \( A \setminus B \) is zero.

It is known that the essential closure of \( A \) is \( \sigma (C^*) \). This implies the essential closure of \( B \) is \( \sigma (C^*) \). Thus whenever \( \Delta \) is a disc intersecting \( \sigma (C) \) there are points in \( \Delta \) at which \( \int_0^\infty (\delta_z (t)/t) \ dt = \infty \). Consequently, if \( \Delta \) is a disc intersecting \( \sigma (C) \setminus \pi_0 (C) \) there are points \( z \) in \( \Delta \) with

\[
\det \left[ I_N - \left[ (\tilde{\psi}_i (z), \tilde{\psi}_j (z)) \right]_{N \times N} \right] = 0. \quad (5)
\]

It will be shown that the matrix \( R (z) = [(\tilde{\psi}_i (z), \tilde{\psi}_j (z))] \) satisfies \( 0 < R (z) < I_N \). Note first

\[
\tilde{\psi}_i (z) = W^*_z (P_z + D)^{-1/2} \psi_i \quad (1 < i < N).
\]

Therefore
\[
\left( \hat{\psi}_i(z), \hat{\psi}_j(z) \right) = \left( (P_z + D)^{-1/2}\psi_i, (P_z + D)^{-1/2}\psi_j \right)
\]
\[
= \lim_{\lambda \to 0} \left( (P_z + D - \lambda)^{-1/2}\psi_i, (P_z + D - \lambda)^{-1/2}\psi_j \right)
\]
\[
= \lim_{\lambda \to 0} \left( (P_z + D - \lambda)^{-1}\psi_i, \psi_j \right).
\]

The result will follow when it is established that
\[
R(z, \lambda) = \left[ \left( (P_z + D - \lambda)^{-1}\psi_i, \psi_j \right) \right] < I_N.
\]

Let \( x = [x_1, \ldots, x_N] \in C^N \). Then
\[
\langle R(z, \lambda)x, x \rangle = \left( (P_z + D - \lambda)^{-1}\omega, \omega \right)
\]
where \( \omega = \sum_{i=1}^{N} \overline{x_i}\psi_i \). Since \( (P_z + D - \lambda)^{-1} < (D - \lambda)^{-1} \) we have
\[
\langle R(z, \lambda)x, x \rangle < \left( (D - \lambda)^{-1}\omega, \omega \right).
\]

Using the fact that \( D = \sum_{i=1}^{N} \psi_i \otimes \psi_i \) with \( \psi_i \) orthogonal one computes
\[
(D - \lambda)^{-1}\omega = \sum_{i=1}^{N} \frac{\overline{x_i}}{\|\psi_i\|^2 - \lambda} \psi_i.
\]

Therefore
\[
\langle R(z, \lambda)x, x \rangle < \sum_{i=1}^{N} \frac{|x_i|^2\|\psi_i\|^2}{\|\psi_i\|^2 - \lambda} \leq \sum_{i=1}^{N} |x_i|^2.
\]

This shows \( R(z, \lambda) < I_N \) and the estimate \( R(z) < I_N \) follows.

Note that if \( e_1, \ldots, e_N \) are orthonormal in \( \mathcal{K} \), then with \( A(z) = \sum_{i=1}^{N} (f, e_i)\hat{\psi}_i(z) \), the matrix \( R(z) \) is (unitarily equivalent to) the product \( A^*(z)A(z) \).

Suppose now that \( \Delta \cap [\sigma(C) \setminus \pi_0(C)] = \emptyset \) and \( \hat{\psi}_1, \ldots, \hat{\psi}_N \) are analytic on \( \Delta \). Then from (5) it follows that the norm of \( A(z) \) is achieved at \( z_0 \in \Delta \). This means for some nonzero \( f \),
\[
\|A(z_0)f\| = 1 = \max_{z \in \Delta} \|A(z)\|.
\]

It follows easily from the maximum modulus theorem that \( A(z)f \) is constant on \( \Delta \).

This yields \( \sum_{i=1}^{N} (f, e_i)\hat{\psi}_i(z) \) is a constant vector \( g \) on \( \Delta \). Let \( h = \sum_{i=1}^{N} (f, e_i)\hat{\psi}_i \). Then \( (C - z)g = h (z \in \Delta) \) which is impossible. We conclude that one of \( \hat{\psi}_1, \ldots, \hat{\psi}_N \) is not analytic on \( \Delta \).

The proof of Theorem 2 can now be completed as follows. Let \( \Delta \cap \sigma(C) \neq \emptyset \). If \( \lambda \in \Delta \cap \pi_0(C) \), then any eigenvector \( f_\lambda \) associated with \( \lambda \) satisfies \( \sigma_C(f_\lambda) = \{\lambda\} \subset \Delta \). If no such \( \lambda \) exists then one of the weakly continuous functions \( \hat{\psi}_1, \ldots, \hat{\psi}_N \) say \( \hat{\psi}_\varphi \) fails to be analytic on \( \Delta \). This means that for some simple closed curve \( \Gamma \subset \Delta \), \( \varphi = \int_{\Gamma} \hat{\psi}_\varphi(x) \) does not exist. As in Stampfli [7] one shows \( \sigma_C(\varphi) \subset \Delta \). This ends the proof of Theorem 2.
References


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