HYPOELLIPTICITY IN THE COMPLEXES
OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Sufficient conditions for the pseudodifferential operator defined on a complex to be hypoelliptic are investigated.

Introduction. In this paper we shall continue the study of hypoellipticity in the complexes of pseudodifferential operators introduced in Treves [3]. In [3], Treves characterized the necessary and sufficient condition of the hypoellipticity of the pseudodifferential operators in the complexes in the 0-dimensional level. We shall study the case when dimension p (i.e., pth step in the complex) is greater than 0.

In the first section we derive a plausible definition of hypoellipticity in the complexes of pseudodifferential operators in the $p > 0$ dimensional level. In the second section we give certain sufficient conditions for the pseudo-differential operators to be hypoelliptic, while in the third section we discuss the case when the pseudodifferential operators are nondegenerate.

1. Hypoellipticity in the complexes of pseudodifferential operators. Let R^ν (resp. R^n) be a ν-dimensional (resp. n-dimensional) Euclidean space. Throughout this paper, we shall denote by Ω a fixed open subset of R^ν. For any real number s we denote by $H^s = H^s(R^n)$ the standard global Sobolev space on R^n, i.e., the space of tempered distributions u on R^n whose Fourier transforms $\hat{u}$ are measurable functions on R^n satisfying

$$|u|_s = (2\pi)^{-n}\left\{ \int \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi \right\}^{1/2} < +\infty.$$

We shall set

$$H^{-\infty} = \bigcup_{s \in R} H^s, \qquad H^\infty = \bigcap_{s \in R} H^s.$$

When E is either $H^{-\infty}$ or H^∞, we shall denote by $\Lambda^p C^\infty(\Omega; E)$ the sheaf of germs of C^∞ p-forms in Ω, valued in E and by $\Lambda^p \mathcal{D}'(\Omega; E)$ the sheaf of germs of E-valued p-currents in Ω.

Let $b(t, \xi) = \sum_{j=1}^\nu b_j(t, \xi) dt_j$ be a C^∞ one-form in Ω depending on the parameter ξ of R_n, the dual of R^n, where $b_j(t, \xi)$ is a pseudodifferential operator from E into E. We shall then form a differential-pseudodifferential

operator

$$D = d_t + b(t, D_x)\Lambda$$

where Λ is the exterior multiplication on E. This enables us to define a linear operator

$$D^p: \Lambda^p C^\infty(\Omega; E) \to \Lambda^{p+1} C^\infty(\Omega; E)$$

for each $p = 0, 1, \ldots, \nu - 1$.

We shall assume that $b(t, \xi) = d_t B(t, \xi)$ where $B(t, \xi)$ is real valued and positive-homogeneous of degree one in ξ-variable.

Now let us consider the complex of the sheaves

$$\ldots \Lambda^p C^\infty(\Omega; H^{-\infty}) \overset{D^p}{\to} \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}) \to \ldots.$$

We note that for any $u \in \Lambda^p C^\infty(\Omega; H^{-\infty})$ and any $f \in \Lambda^{p+1} C^\infty(\Omega; H^{-\infty})$, $Du = f$ and $f \in \Lambda^{p+1} C^\infty(\Omega; H^\infty)$ does not imply, in general, $u \in \Lambda^p C^\infty(\Omega; H^\infty)$. In fact, there can be an element $v \in \Lambda^p C^\infty(\Omega; H^{-\infty})$ such that $Dv = 0$ but $v \notin \Lambda^p C^\infty(\Omega; H^\infty)$ (cf. [3, Theorem III.1.1]). Therefore, with more generality, when $p > 0$, we think of the following problem; under which conditions, if $Du = f$ and $f \in \Lambda^{p+1} C^\infty(\Omega; H^{-\infty})$, there exists $v \in \Lambda^{p-1} C^\infty(\Omega; H^{-\infty})$ such that $u - Dv \in \Lambda^p C^\infty(\Omega; H^\infty)$.

We recall that the space of pth microfunctions are, by definition,

$$\Lambda^p C^\infty(\Omega; H^{-\infty}) / \Lambda^p C^\infty(\Omega; H^\infty).$$

Then our complex $\{\Lambda^p C^\infty(\Omega; H^{-\infty}); D\}$ induces in a natural way, a complex of micro-functions, namely,

$$\ldots \to \Lambda^p C^\infty(\Omega; H^{-\infty}) / \Lambda^p C^\infty(\Omega; H^\infty)$$

$$\overset{\dot{D}^p}{\to} \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}) / \Lambda^{p+1} C^\infty(\Omega; H^\infty) \to \ldots.$$

Then the previous problem is precisely the problem: when $p > 0$, under which conditions $\dot{D}\dot{u} = 0$ implies $\dot{u} \in \operatorname{Im} \dot{D}$ for any pth microfunction $\dot{u}$. Under this motivation, we adopt the following definition.

DEFINITION 1.1. D is hypoelliptic in the dimension p ($p = 1, 2, \ldots, \nu - 1$) if and only if $\operatorname{Im} \dot{D}^{p-1} = \ker \dot{D}^p$.

We note that the above definition is a true generalization of the hypoellipticity defined when $p = 0$ in [3] if we set $\Lambda^{-1} C^\infty(\Omega; E) = 0$ for $E = H^{-\infty}$ or H^∞.

The necessary and sufficient conditions for the hypoellipticity in this case are given in [3].

We also remark that, in the above whole arguments, $\Lambda^p C^\infty(\Omega; H^{-\infty})$ may be replaced by $\Lambda^p \mathcal{D}'(\Omega; H^{-\infty})$ but keeping $\Lambda^p C^\infty(\Omega; H^\infty)$ as the same. However, we would not have any gain in generality by this replacement (cf. [3, Proposition III.2.2]).

2. Sufficient conditions for the hypoellipticity. Let the complex $\{\Lambda^p C^\infty(\Omega; H^{-\infty}); D\}$ be given as in §1.

When U is an open subset of R^ν, we have pth DeRham homology group $H_p(U)$ in U induced by the complex

$$\cdots \to \Lambda^p C_c^\infty(U) \xrightarrow{d} \Lambda^{p+1} C_c^\infty(U) \to \cdots$$

with the usual exterior derivative d. Let U and V be open subsets of R^ν. Let θ, θ' be two open subsets of U such that $\theta' \subset \theta \cap V$. For any $\xi \in R_n$ and for any real number γ, we write

$$\theta(\xi, \gamma) = \{ t \in \theta; B(t, \xi) < \gamma \}$$

and similarly with θ' substituted for θ. The natural injections of $\theta'(\xi, \gamma)$ into V and into $\theta(\xi, \gamma)$, respectively, give rise to two homomorphisms

$$H_p(\theta'(\xi, \gamma)) \underset{j_p}{\overset{i_p}{\longrightarrow}} \begin{array}{c} H_p(V) \\ H_p(\theta(\xi, \gamma)) \end{array}$$

We introduce a property Ψ_p as follows:

(Ψ_p) For every point t_0 in Ω and to every open neighborhood U of t_0, there is another open neighborhood V of t_0, $V \subset U$, such that for every open subset $\theta' \subset V$, there is a relatively compact open subset θ of U containing θ' such that for any $\xi \in R^n$ and for any real number γ

$$\operatorname{Ker} i_p \subset \operatorname{Ker} j_p.$$

THEOREM 2.1. *If Ω has the property Ψ_p and Ψ_{p-1} ($p > 0$), then D is hypoelliptic in the dimension p.*

PROOF. We note that our operator D can be extended in a natural way (cf. [3]) as a linear operator

$$D^p: \Lambda^p \mathcal{D}'(\Omega; \mathrm{FLoc}^2) \to \Lambda^{p+1} \mathcal{D}'(\Omega; \mathrm{FLoc}^2)$$

where FLoc^2 are the space of generalized functions on R^n whose Fourier transforms are locally L^2-functions on R_n. Taking Fourier transform we have a linear operator

$$\hat{D}^p: \Lambda^p \mathcal{D}'(\Omega; \mathrm{Loc}^2) \to \Lambda^{p+1} \mathcal{D}'(\Omega; \mathrm{Loc}^2).$$

When E is either $H^{-\infty}$ or H^∞, we set $B_D^{p+1} C^\infty(\Omega; E)$ the space of elements $f \in \Lambda^p C^\infty(\Omega; E)$ such that for almost every $\xi \in R_n$, the $p + 1$ form $e^{B(t,\xi)} \hat{f}(\xi, t)$ is a coboundary in the complex $\{ \Lambda^p \mathcal{D}'(\Omega; \mathrm{Loc}^2); \hat{D}^p \}$.

Now let f be a germ of a $p + 1$ form at $t_0 \in \Omega$ such that $Du = f$ for some $u \in \Lambda^p(C^\infty; H^{-\infty})$. We may take a neighborhood U of t_0 to be homologically trivial and may assume f is defined on U as a $p + 1$ form. Since $Df = 0$, f belongs to $B_D^{p+1} C^\infty(U, H^\infty)$ since $B_D^{p+1} C^\infty(U; E)$ is equal to the kernel of $D: \Lambda^p C^\infty(U; E) \to \Lambda^{p+1} C^\infty(U; E)$ when E is either $H^{-\infty}$ or H^∞ and when U is homologically trivial (cf. [3, II.4]). Since Ω has the property Ψ_p, we may choose a neighborhood V of t_0 satisfying the condition Ψ_p. Then using the [3, Theorem II.1.2], we conclude that, for any relatively compact open subset W

of V, $Dw = f$ has a solution $w \in \Lambda^p C^\infty(W; H^\infty)$ in W.

Since $Du = f$ and $Dw = f$, we have

$$D(u - w) = f \quad \text{in } W.$$

Note that $u - w \in \Lambda^p C^\infty(W; H^{-\infty})$.

We may take W to be homologically trivial so that $u - w \in B_D^p C^\infty(W; H^{-\infty})$. Since Ω has the property Ψ_{p-1}, using [3, Theorem II.1.2] again, there exists $v \in \Lambda^{p-1} C^\infty(W'; H^{-\infty})$ for some neighborhood W' of t_0 such that $Dv = u - w$. This shows that $u - Dv \in \Lambda^p C^\infty(\Omega; H^\infty)$, thus completing our proof.

3. The nondegenerate case. In this section let us assume that $B(t, \xi)$ satisfies that $\forall \xi \in R_n \setminus \{0\}$ every critical point of $B(t, \xi)$ in Ω is nondegenerate, i.e., if $d_t B(t, \xi) = 0$, then $\det \partial_t^2 B(t, \xi) \neq 0$ for any $t \in \Omega$ and $\xi \in R_n \setminus \{0\}$. $\partial_t^2 B(t, \xi)$ stands for the Hessian matrix of $B(t, \xi)$ with respect to t. The index of $B(t, \xi^0)$ at t^0 is, by definition, the number $\mu = \mu(t^0, \xi^0)$ of strictly negative eigenvalues of $\partial_t^2 B(t^0, \xi^0)$.

DEFINITION 3.1. We say that for $p = 1, \ldots, \nu$ the property $Z(p)$ holds for D on Ω if $\forall \xi \in R_n \setminus \{0\}$, the index of $B(t, \xi)$ at every critical point t of $B(\cdot, \xi)$ in Ω is not equal to p.

THEOREM 3.1. *Suppose that, for all $\xi \in R_n \setminus \{0\}$, every critical point of the function $B(\cdot, \xi)$ in Ω is nondegenerate. Let p be an integer such that $0 < p \leqslant \nu$. Then D is hypoelliptic in the dimension p if $Z_{(p)}$ and $Z_{(p+1)}$ hold for D on Ω.*

PROOF. Let $f \in \Lambda^{p+1} C^\infty(\Omega; H^\infty)$ such that $Du = f$ for some $u \in \Lambda^p C^\infty(\Omega; H^{-\infty})$. Since $Z_{(p+1)}$ holds for D and $Df = 0$, using [3, Theorem II.4.3]. There exists $w \in \Lambda^p C^\infty(\Omega; H^\infty)$ such that $Dw = f$. Since $Du = f$, $D(u - w) = 0$. Since $Z_{(p)}$ holds for D and $D(u - w) = 0$, using [3, Theorem II.4.3] again, we have $v \in \Lambda^{p-1} C^\infty(\Omega; H^{-\infty})$ such that $Dv = u - w$. Thus $u - Dv \in \Lambda^p C^\infty(\Omega; H^\infty)$, completing the proof.

COROLLARY 1. *With the same hypothesis as in the theorem, let E be either $H^{-\infty}$ or H^∞. If the complex $\{\Lambda^p C^\infty(\Omega; E); D\}$ is globally locally solvable in the p- and $(p + 1)$-dimensional levels, then D is hypoelliptic in the dimension p.*

PROOF. $Z_{(p)}$ (resp. $Z_{(p+1)}$) is equivalent to the global local solvability of the complex $\{\Lambda^p C^\infty(\Omega; E); D\}$ at the p-dimensional (resp. $(p + 1)$-dimensional) level. Thus the previous theorem completes our proof.

COROLLARY 2. *With the same hypothesis as in the theorem, we assume further that D is antipodal, i.e., $B(t, -\xi) = -B(t, \xi)$ for every $\xi \in R_n \setminus \{0\}$. Then D is hypoelliptic in the dimension p if $Z_{(p)}$, $Z_{(\nu - p)}$, $Z_{(p+1)}$ and $Z_{(\nu - p - 1)}$ hold for D on Ω.*

PROOF. If D is antipodal, then the symmetry $\xi \to -\xi$ changes the index μ of $B(t, \xi)$ of an arbitrary critical point of $B(\cdot, \xi)$ into $\nu - \mu$. Thus applying the previous theorem completes the proof.

We note that the ν-dimensional induced boundary complex introduced in [2] can be transformed by the Fourier integral operator modulo smooth operators to the D complex on R^ν with $n = \nu + 1$ and $b_j(t, \xi) = \pm t_j|\xi|$ (cf. [1]). Since the boundary operator ∂_b is antipodal, we have the following corollary.

COROLLARY 3. *Let*

$$\ldots \to B^{0,p} \overset{\partial_b}{\to} B^{0,p+1} \to \ldots$$

be the ν-dimensional induced boundary complex. Then ∂_b is hypoelliptic in the dimension $p > 0$ in the sense analogous to Definition 1.1 if the Levi matrix at the critical point of ∂_b has no index identical to $p, \nu - p, p + 1, \nu - p - 1$.

PROOF. Since the index of the Levi matrix of ∂_b at (t, ξ) is the same as the index of the Hessian matrix at (t, ξ), the corollary follows from the previous theorem and [3, Proposition III.2.2] in p-dimensional form.

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