PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 3, December 1978

LATTICE PROPERTIES OF INTEGRAL OPERATORS

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ABSTRACT. In this paper we are concerned with linear operators K: $L \to M$, where L is a Riesz subspace of measurable, finite a.e. functions and M is the class of all measurable, finite a.e. functions defined by

$$K(f)(x) = \int k(x,y)f(y)dy$$

where $k(x,y)$ is a measurable kernel. It will be shown that the class $I[L, M]$ of all such integral operators is a Dedekind complete Riesz space, an ideal and a band in the space of order bounded linear maps T: $L \to M$.

Definitions and notation.

Measure. For $i = 1, 2$, let (X_i, A_i, μ_i) be a σ-finite measure space with nonnegative measure i. (X_1, A_1, μ_1) is assumed to be a separable measure space: see [7, p. 69]. M_i is the collection of equivalence classes of A_i measurable, finite μ_i a.e. functions modulo μ_i null functions. We take $(X_2 \times X_1, A_2 \times A_1, \mu_2 \times \mu_1)$ to be the completed product space and M_{21} the collection of equivalence class of $\mu_2 \times \mu_1$ measurable, finite a.e. functions modulo $\mu_2 \times \mu_1$ null functions. By $\overline{M}_2$ we shall mean the collection of measurable, real valued, finite everywhere functions defined on X_2; ψ: $\overline{M}_2 \to M_2$ is the canonical homomorphism that sends $f \in \overline{M}_2$ to its equivalence class $\{f\} \in M_2$. We will write $\int f(t)dt$ to mean the usual Lebesgue integral $\int f(t)d\mu(t)$ of f relative to the measure μ. For a measurable set E, $\chi(E)$ will denote the characteristic function of E.

Order structure. In this paper we shall follow [3] and [5] in matters related to Riesz space theory. Let L be a Riesz subspace and an ideal of M_1 for which there exists a sequence $x_j \in A_1$ of finite μ_1 measure such that $x_j \subset x_{j+1}$, $\bigcup_j x_j = X_1$, and $\chi(x_j) \in L$. For a given L such a sequence will be called admissible. OB$[L, M_2]$ will denote the linear maps T: $L \to M_2$ which are order bounded: see [5]. If K: $L \to M_2$ and there exists $k \in M_{21}$ such that for all $f \in L$,

$$K(f)(x) = \int k(x,y)f(y)d\mu_1(y)$$

we shall call K an integral operator and both abbreviate and denote this relationship by $K = [k]$. Since $|K(f)| \leqslant \int |k(x,y)f(y)|dy < \infty$ μ_2 a.e., $K \in$ OB$[L, M_2]$. By $I[L, M_2]$ we shall mean the collection of all integral operators K: $L \to M_2$. It is well known that OB$[L, M_2]$ is a Dedekind complete Riesz space in which every operator $T = T_1 - T_2$ where T_1 and T_2 are positive

Received by the editors November 29, 1977 and, in revised form, April 12, 1978.
AMS (MOS) subject classifications (1970). Primary 47A65, 47B55, 47D15.

operators: see **[5]**. By ker$[L, M]$ we shall mean the collection of $k \in M_{21}$ such that for all $f \in L$, $\int k(x,y)f(y)dy < \infty$ μ_2 a.e. Clearly each such k defines an operator $K \in I[L, M_2]$. ker$[L, M_2]$ is obviously a linear subspace of M_{21} and the definition of the Lebesgue integral implies that if $k \in \ker[L, M_2]$, then $|k| \in \ker[L, M_2]$. Simple calculation shows that ker$[L, M_2]$ is a Riesz subspace and an ideal of M_{21}: see **[2]** and **[11]**.

For $x_n, x \in R$, a Riesz space, we write $x_n \to x(0)$ when x_n converges to x in order, $0 \leqslant x_n \uparrow x$ when $x_n \leqslant x_{n+1}$ and $\sup x_n = x$, $0 \leqslant x_n \downarrow 0$ when $0 \leqslant x_n \geqslant x_{n+1}$ and $\inf x_n = 0$. If $x_n \to x(0)$ then there exists $y_n \in R$ such that $0 \leqslant y_n \downarrow 0$ and $|x_n - x| \leqslant y_n$.

If R_i is a Riesz subspace of M_i then for $f_n, f \in R_i$, $f_n \to f(0)$ in R_i is equivalent to $f_n(x) \to f(x)$ μ_1 a.e. x and there exists $f_0 \in R_i$ such that $|f_n| \leqslant |f_0|$. Also $f_n \to f(0)M_i$ if and only if $f_n(x) \to f(x)$ μ_i a.e. A map T: $R_1 \to R_2$ is order continuous whenever T maps order convergent sequences to order convergent sequences.

THEOREM 1. *$I[L, M_2]$ is a Riesz space where $|[k]| = [|k|]$ and $[k_1] \vee [k_2] = [k_1 \vee k_2]$.*

This theorem was proved concurrently and independently by **[1]** and **[2]**. It follows that for $K \in I[L, M_2]$, where $K = [k]$, if $0 \leqslant K$, then $0 \leqslant k$ $\mu_2 \times \mu_1$ a.e., and $K = 0$ if and only if $k = 0$ $\mu_2 \times \mu_1$ a.e. Thus the map $w([k]) = k$ is a Riesz isomorphism of $I[L, M_2]$ onto ker$[L, M_2]$: see **[3**, p. 98]. Theorem 3 will show w to be a normal Riesz isomorphism: i.e. one that preserves arbitrary suprema: see **[3**, p. 103].

THEOREM 2. *$I[L, M_2]$ is an ideal of* OB$[L, M_2]$. *Specifically if $T \in$* OB$[L, M_2]$, *$K \in I[L, M_2]$, and $0 \leqslant T \leqslant K$, then there exists $h \in$* ker$[L, M_2]$ *such that $T = [h]$.*

For the proof of this theorem we require the theorem from **[4]** which we shall state as Lemma. For a map T: $L \to M_2$ and S an ideal of L we say that $\hat{T}$: $S \to \overline{M}_2$ is a lift of T on S when for all $f \in S$, $\psi \circ \hat{T}(f) = T(f)$.

LEMMA. *If T: $L \to M_2$ is a positive linear map, then there exists $h \in$* ker$[L, M_2]$ *such that $T = [h]$ if and only if*

(A) *T: $L \to M_2$ is order continuous.*

(B) *For each j there exists a lift of T on $L_\infty(x_j)$, $\hat{T}$: $L_\infty(x_j) \to \overline{M}_2$ which is positive linear and order continuous.*

We remark that $0 \leqslant T$: $L \to M_2$ order continuous is equivalent to $0 \leqslant f_n \downarrow 0$ implies $T(f_n) \to 0$ a.e.: see **[5**, p. 214].

Since (X_1, A_1, μ_1) is a separable measure space, the relativized space $(x_j, A_1 \cap x_j, \mu_1)$ is separable also. Let Ω_j be a countable subset of $A_1 \cap x_j$ that is dense in the metric $d(E, F) = \mu_1(E \bigtriangleup F)$. Let S_j be a maximal linearly independent subcollection of all finite linear combinations of $\chi(E)$ where

$E \in \Omega_j$ with rational number coefficients. Clearly S_j is countable: $S_j = \{f_i\}_i$. For each i choose $g_i \in \{\overline{T(f_i)}\}$: i.e. $\psi(g_i) = T(f_i)$. Now define $\hat{T}$: $S_j \to \overline{M}_2$ by $\hat{T}(f_i) = g_i$. $\hat{T}$ is a lift of T on S_j. Our objective is to extend the lift $\hat{T}$ to $L_\infty(x_j)$ in such a manner that $0 \leqslant \hat{T}(f) \leqslant \hat{K}(f)$ is true for $f \in L_\infty(x_j)$.

Since K is an integral operator we know, by the above lemma, that there exists a lift $\hat{K}$ of K on $L_\infty(x_j)$. The following statements are true for a.e. x.

(1) For $f_1, f_2 \in S_j$ and r rational,

$$\hat{T}(rf_1 + f_2)(x) = r\hat{T}(f_1)(x) + \hat{T}(f_2)(x) \quad \text{a.e. } x,$$

(2) for $0 \leqslant f \in S_j$, $0 \leqslant \hat{T}(f)(x) \leqslant \hat{K}(f)(x)$ a.e. x,

(3) for $f \in S_j$ and $f = 0$ μ_1 a.e., then $\hat{T}(f)(x) = 0$ a.e. x.

Because S_j is countable, there are only countably many statements of the form (1), (2) and (3). Thus there exists $A_j \in A_2$ such that $\mu_2(A_j) = 0$ and (1), (2) and (3) are true for $x \notin A_j$. We shall redefine the lift $\hat{T}$ on S_j by taking for $x_0 \in A_j$, $\hat{T}(f_i)(x_0) = 0$ for all $f_i \in S$. The result is a "new" lift of T or S_j which differs from the previous lift on A_j only, a μ_2 null set. Thus we may now take statements (1), (2) and (3) to be true for all $x \in X_2$.

To extend $\hat{T}$ to $L_\infty(x_j)$ we define the functional F_x: $S_j \to$ reals by $F_x(f) = \hat{T}(f)(x)$ and apply the extension procedure [5, X.5.1]. In the terminology of [5], S_j is a linear sublattice of $L_\infty(x_j)$ that majorizes it, and F_x is a positive linear functional which is "order continuous with respect to convergence in $L_\infty(x_j)$" by statement (2) and Lemma X.5.1. That $L_\infty(x_j)$ is the Borel superstructure over S_j follows from the fact that order limits of members of S_j in $L_\infty(x_j)$ include linear combinations of characteristic functions $\chi(E)$ for arbitrary $E \in X_j \cap a_1$, and order limits in $L_\infty(x_j)$ of these simple functions include $f \in L_\infty(x_j)$. That (1), (2) and (3) remain true for all $x \in X_2$ and $f \in L_\infty(x_j)$ also follows from the above remarks on order limits of sequences and repeated applications of the inequality

$$|\hat{T}(f_n)(x) - \hat{T}(f_m(x))| \leqslant \hat{T}(|f_n - f_m|)(x) \leqslant \hat{K}(|f_n - f_m|)(x).$$

For $f \in L_\infty(x_j)$, the measurability of $\hat{T}(f)$ follows from the fact that $\hat{T}(f)$ is the pointwise limit of measurable functions.

Once the lift $\hat{T}$: $L_\infty(x_j) \to \overline{M}_2$ that satisfies (1), (2) and (3) has been established, one applies Lemma 1. Condition (B) of Lemma 1 results from (2). Thus there exists $h \in \ker[L, M_2]$ such that $H = [h]$. □

THEOREM 3. *$I[L, M_2]$ is a band in* OB$[L, M_2]$.

PROOF. In order to show that the ideal $I[L, M_2]$ is a band it is sufficient to show that for any collection $\{K_\alpha\} \subset I[L, M_2]$ such that $0 \leqslant K_\alpha \leqslant T$, where $T \in \mathrm{OB}[L, M_2]$, we have $\sup_\alpha K_\alpha = K \in I[L, M_2]$. Let $K_\alpha = [k_\alpha]$. In the terminology of [5], M_{21} is a K^+ space. If $\{k_\alpha\}_\alpha$ is bounded in M_{21}, then there exists a countable subset $\{k_{\alpha(i)}\}_i$ such that $\sup_\alpha k_\alpha = \sup_i k_{\alpha(i)}$: see [5, VI.1.1]. If $\{k_\alpha\}_\alpha$ is unbounded, then there also exists a countable unbounded subset: [5, VI.6.3]. In either case let us denote $h = \sup_\alpha k\alpha = \sup_i k_\alpha(i)$ where $\{k_{\alpha(i)}\}_i$

is an appropriately chosen countable subcollection. Let $h_m = \sup\{k_{\alpha(i)}: 1 \leqslant i \leqslant n\}$, then $h_n \in \ker[L, M_2]$, $0 \leqslant h_n \uparrow h$, and $[h_n] \leqslant T$. For $0 \leqslant f \in L$ we have $0 \leqslant \int h_n f\, dy \leqslant T(f)$. By an application of the monotone convergence theorem, we have $0 \leqslant \int hf \leqslant T(f)$. Thus $\int(\sup_\alpha k_\alpha) f\, dy < \infty$ a.e. x. Since f was arbitrary in L, $\sup_\alpha k_\alpha = k < \infty$ $\mu_2 \times \mu_1$ a.e. and $k \in \ker[L, M_2]^+$. Since $[k_\alpha] \leqslant [k]$, it follows that $\sup_\alpha[k_\alpha] \leqslant [k]$ and by Theorem 2

$$\sup_\alpha[k_\alpha] = H \in I[L, M_2].$$

Let $H = [h]$. Since $0 \leqslant h_n \uparrow k$ and $0 \leqslant h_n \leqslant h$, we must have $k \leqslant h$. Since $H \leqslant K$ we have $0 \leqslant h \leqslant k$. Thus $h = k$ and $\sup_\alpha[k_\alpha] = [\sup_\alpha k_\alpha]$. $\square$

It should be noted that Theorem 3 can be found in a less general context in [**10**]. However, with the use of [**10**] and the lemma from [**4**, Theorem 3] can be obtained easily.

Part of this paper was completed in fulfillment of the Ph.D. degree at the University of Southern California in 1971. I wish to gratefully acknowledge the help and encouragement of my thesis advisor Professor Alan Schumitzky. I also wish to thank Professor W. A. J. Luxemburg of the California Institute of Technology for his several very helpful suggestions.

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