LATTICE PROPERTIES OF INTEGRAL OPERATORS

LAWRENCE LESSNER

Abstract. In this paper we are concerned with linear operators \( K : L \to M \), where \( L \) is a Riesz subspace of measurable, finite a.e. functions and \( M \) is the class of all measurable, finite a.e. functions defined by

\[
K(f)(x) = \int k(x, y) f(y) \,dy
\]

where \( k(x, y) \) is a measurable kernel. It will be shown that the class \( I[L, M] \) of all such integral operators is a Dedekind complete Riesz space, an ideal and a band in the space of order bounded linear maps \( T : L \to M \).

Definitions and notation.

Measure. For \( i = 1, 2 \), let \( (X_i, A_i, \mu_i) \) be a \( \sigma \)-finite measure space with nonnegative measure \( i \). \( (X_1, A_1, \mu_1) \) is assumed to be a separable measure space: see [7, p. 69]. \( M_i \) is the collection of equivalence classes of \( A_i \) measurable, finite \( \mu_i \) a.e. functions modulo \( \mu_i \) null functions. We take \( (X_2 \times X_1, A_2 \times A_1, \mu_2 \times \mu_1) \) to be the completed product space and \( M_2 \) the collection of equivalence class of \( \mu_2 \times \mu_1 \) measurable, finite a.e. functions modulo \( \mu_2 \times \mu_1 \) null functions. By \( M_2 \) we shall mean the collection of measurable, real valued, finite everywhere functions defined on \( X_2 \); \( \psi : M_2 \to M_2 \) is the canonical homomorphism that sends \( f \in M_2 \) to its equivalence class \( \{f\} \in M_2 \). We will write \( \int f(t) \,dt \) to mean the usual Lebesgue integral \( \int f(t) \,d\mu(t) \) of \( f \) relative to the measure \( \mu \). For a measurable set \( E \), \( \chi(E) \) will denote the characteristic function of \( E \).

Order structure. In this paper we shall follow [3] and [5] in matters related to Riesz space theory. Let \( L \) be a Riesz subspace and an ideal of \( M_1 \) for which there exists a sequence \( x_j \in A_1 \) of finite \( \mu_1 \) measure such that \( x_j \subset x_{j+1} \), \( \bigcup_j x_j = X_1 \), and \( \chi(x_j) \in L \). For a given \( L \) such a sequence will be called admissible. \( OB[L, M_2] \) will denote the linear maps \( T : L \to M_2 \) which are order bounded: see [5]. If \( K : L \to M_2 \) and there exists \( k \in M_2 \) such that for all \( f \in L \),

\[
K(f)(x) = \int k(x, y) f(y) \,d\mu_1(y)
\]

we shall call \( K \) an integral operator and both abbreviate and denote this relationship by \( K = [k] \). Since \( |K(f)| \leq \int |k(x, y) f(y)| \,dy < \infty \) a.e., \( K \in OB[L, M_2] \). By \( I[L, M_2] \) we shall mean the collection of all integral operators \( K : L \to M_2 \). It is well known that \( OB[L, M_2] \) is a Dedekind complete Riesz space in which every operator \( T = T_1 - T_2 \) where \( T_1 \) and \( T_2 \) are positive.
operators: see [5]. By \( \ker[L, M] \) we shall mean the collection of \( k \in M_{21} \) such that for all \( f \in L, \int k(x, y)f(y)dy < \infty \mu_2 \) a.e. Clearly each such \( k \) defines an operator \( K \in I[L, M_2]. \) \( \ker[L, M_2] \) is obviously a linear subspace of \( M_{21} \) and the definition of the Lebesgue integral implies that if \( k \in \ker[L, M_2], \) then \( |k| \in \ker[L, M_2]. \) Simple calculation shows that \( \ker[L, M_2] \) is a Riesz subspace and an ideal of \( M_{21} \): see [2] and [11].

For \( x_n, x \in \mathcal{R}, \) a Riesz space, we write \( x_n \rightarrow x(0) \) when \( x_n \) converges to \( x \) in order, \( 0 < x_n \uparrow x \) when \( x_n < x_{n+1} \) and sup \( x_n = x, \) \( 0 < x_n \downarrow 0 \) when \( 0 < x_n > x_{n+1} \) and inf \( x_n = 0. \) If \( x_n \rightarrow x(0) \) then there exists \( y_n \in \mathcal{R} \) such that \( 0 < y_n \downarrow 0 \) and \( |x_n - x| < y_n. \)

If \( R_i \) is a Riesz subspace of \( M_i \) then for \( f_n, f \in R_i, \) \( f_n \rightarrow f(0) \) in \( R_i \) is equivalent to \( f_n(x) \rightarrow f(x) \mu_1 \) a.e. \( x \) and there exists \( f_0 \in R_i \) such that \( |f_n| < |f_0|. \) Also \( f_n \rightarrow f(0)M_i \) if and only if \( f_n(x) \rightarrow f(x) \mu_i \) a.e. A map \( T: R_1 \rightarrow R_2 \) is order continuous whenever \( T \) maps order convergent sequences to order convergent sequences.

**Theorem 1.** \( I[L, M_2] \) is a Riesz space where \( \|k\| = |\|k\|| \) and \( [k_1] \lor [k_2] = [k_1 \lor k_2]. \)

This theorem was proved concurrently and independently by [1] and [2]. It follows that for \( K \in I[L, M_2], \) where \( K = [k], \) if \( 0 \leq K, \) then \( 0 < k \mu_2 \times \mu_1 \) a.e., and \( K = 0 \) if and only if \( k = 0 \mu_2 \times \mu_1 \) a.e. Thus the map \( \omega([k]) = k \) is a Riesz isomorphism of \( I[L, M_2] \) onto \( \ker[L, M_2]: \) see [3, p. 98]. Theorem 3 will show \( \omega \) to be a normal Riesz isomorphism: i.e. one that preserves arbitrary suprema: see [3, p. 103].

**Theorem 2.** \( I[L, M_2] \) is an ideal of \( \text{OB}[L, M_2]. \) Specifically if \( T \in \text{OB}[L, M_2], K \in I[L, M_2], \) and \( 0 < T \leq K, \) then there exists \( h \in \ker[L, M_2] \) such that \( T = [h]. \)

For the proof of this theorem we require the theorem from [4] which we shall state as Lemma. For a map \( T: L \rightarrow M_2 \) and \( S \) an ideal of \( L \) we say that \( \hat{T}: S \rightarrow M_2 \) is a lift of \( T \) on \( S \) when for all \( f \in S, \psi \circ \hat{T}(f) = T(f). \)

**Lemma.** If \( T: L \rightarrow M_2 \) is a positive linear map, then there exists \( h \in \ker[L, M_2] \) such that \( T = [h] \) if and only if

1. \( T: L \rightarrow M_2 \) is order continuous.
2. For each \( j \) there exists a lift of \( T \) on \( L_\infty(x_j), \) \( \hat{T}: L_\infty(x_j) \rightarrow M_2 \) which is positive linear and order continuous.

We remark that \( 0 < T: L \rightarrow M_2 \) order continuous is equivalent to \( 0 < f_n \downarrow 0 \) implies \( T(f_n) \rightarrow 0 \) a.e.: see [5, p. 214].

Since \( (X_1, A_1, \mu_1) \) is a separable measure space, the relativized space \( (x_j, A_1 \cap x_j, \mu_1) \) is separable also. Let \( \Omega_j \) be a countable subset of \( A_1 \cap x_j \) that is dense in the metric \( d(E, F) = \mu_1(E \triangle F). \) Let \( S_j \) be a maximal linearly independent subcollection of all finite linear combinations of \( \chi(E) \) where
For each \( i \) choose \( g_i \in \{ \mathcal{T}(f_i) \} \): i.e. \( \psi(g_i) = T(f) \). Now define \( \hat{T} : S_j \to M_2 \) by \( \hat{T}(f_i) = g_i \). \( \hat{T} \) is a lift of \( T \) on \( S_j \). Our objective is to extend the lift \( \hat{T} \) to \( L_\infty(x_j) \) in such a manner that \( 0 < \hat{T}(f) < \hat{K}(f) \) is true for \( f \in L_\infty(x_j) \).

Since \( K \) is an integral operator we know, by the above lemma, that there exists a lift \( \hat{K} \) of \( K \) on \( L_\infty(x_j) \). The following statements are true for a.e. \( x \).

1. For \( f_1, f_2 \in S_j \) and \( r \) rational,
   \[
   \hat{T}(rf_1 + f_2)(x) = r\hat{T}(f_1)(x) + \hat{T}(f_2)(x) \quad \text{a.e. } x,
   \]
2. For \( 0 < f \in S_j \), \( 0 < \hat{T}(f)(x) < \hat{K}(f)(x) \) a.e. \( x \),
3. For \( f \in S_j \) and \( f = 0 \) \( \mu_j \) a.e., then \( \hat{T}(f)(x) = 0 \) a.e. \( x \).

Because \( S_j \) is countable, there are only countably many statements of the form (1), (2) and (3). Thus there exists \( A_j \in 2^2 \) such that \( \mu_j(A_j) = 0 \) and (1), (2) and (3) are true for \( x \notin A_j \). We shall redefine the lift \( \hat{T} \) on \( S_j \) by taking for \( x_0 \in A_j \), \( \hat{T}(f)(x_0) = 0 \) for all \( f \in S \). The result is a "new" lift of \( T \) or \( S \) which differs from the previous lift on \( A_j \) only, a \( \mu_j \) null set. Thus we may now take statements (1), (2) and (3) to be true for all \( x \in X_j \).

To extend \( \hat{T} \) to \( L_\infty(x_j) \) we define the functional \( F_x : S_j \to \text{reals} \) by \( F_x(f) = \hat{T}(f)(x) \) and apply the extension procedure [5, X.5.1]. In the terminology of [5], \( S_j \) is a linear sublattice of \( L_\infty(x_j) \) that majorizes it, and \( F_x \) is a positive linear functional which is "order continuous with respect to convergence in \( L_\infty(x_j) \)" by statement (2) and Lemma X.5.1. That \( L_\infty(x_j) \) is the Borel superstructure over \( S_j \) follows from the fact that order limits of members of \( S_j \) in \( L_\infty(x_j) \) include linear combinations of characteristic functions \( \chi(E) \) for arbitrary \( E \in X_j \cap a_i \), and order limits in \( L_\infty(x_j) \) of these simple functions include \( f \in L_\infty(x_j) \). That (1), (2) and (3) remain true for all \( x \in X_j \) and \( f \in L_\infty(x_j) \) also follows from the above remarks on order limits of sequences and repeated applications of the inequality

\[
|\hat{T}(f_n)(x) - \hat{T}(f_m)(x)| \leq \hat{T}(|f_n - f_m|(x)) \leq \hat{K}(|f_n - f_m|(x)).
\]

For \( f \in L_\infty(x_j) \), the measurability of \( \hat{T}(f) \) follows from the fact that \( \hat{T}(f) \) is the pointwise limit of measurable functions.

Once the lift \( \hat{T} : L_\infty(x_j) \to M_2 \) that satisfies (1), (2) and (3) has been established, one applies Lemma 1. Condition (B) of Lemma 1 results from (2). Thus there exists \( h \in \ker[L, M_2] \) such that \( H = [h] \). □

**Theorem 3.** \( I[L, M_2] \) is a band in \( \text{OB}[L, M_2] \).

**Proof.** In order to show that the ideal \( I[L, M_2] \) is a band it is sufficient to show that for any collection \( \{ K_a \} \subset I[L, M_2] \) such that \( 0 < K_a < T \), where \( T \in \text{OB}[L, M_2] \), we have \( \sup_a K_a = K \in I[L, M_2] \). Let \( K_a = [k_a] \). In the terminology of [5], \( M_{21} \) is a \( K^+ \) space. If \( \{ k_a \}_a \) is bounded in \( M_{21} \), then there exists a countable subset \( \{ k_{a(0)} \}_i \) such that \( \sup_a k_a = \sup_i k_{a(0)} \); see [5, VI.1.1]. If \( \{ k_a \}_a \) is unbounded, then there also exists a countable unbounded subset: [5, VI.6.3]. In either case let us denote \( h = \sup_a k \alpha = \sup_i k_{a(0)} \) where \( \{ k_{a(0)} \}_i \)
is an appropriately chosen countable subcollection. Let $h_m = \sup\{k_{a(i)}: 1 < i < n\}$, then $h_n \in \ker\{L, M_2\}$, $0 < h_\uparrow h$, and $[h_n] < T$. For $0 < f \in L$ we have $0 < \int h_n f \, dy < T(f)$. By an application of the monotone convergence theorem, we have $0 < \int h f \, dy < T(f)$. Thus $\int (\sup_a k_a) f \, dy < \infty$ a.e. $x$. Since $f$ was arbitrary in $L$, $\sup_a k_a = k < \infty \mu_2 \times \mu_1$ a.e. and $k \in \ker\{L, M_2\}^\perp$. Since $[k_a] < [k]$, it follows that $\sup_a[k_a] < [k]$ and by Theorem 2

$$\sup_a[k_a] = H \in I[L, M_2].$$

Let $H = [h]$. Since $0 < h_\uparrow k$ and $0 < h_n < h$, we must have $k < h$. Since $H < K$ we have $0 < h < k$. Thus $h = k$ and $\sup_a[k_a] = [\sup_a k_a]$. \qed

It should be noted that Theorem 3 can be found in a less general context in [10]. However, with the use of [10] and the lemma from [4, Theorem 3] can be obtained easily.

Part of this paper was completed in fulfillment of the Ph.D. degree at the University of Southern California in 1971. I wish to gratefully acknowledge the help and encouragement of my thesis advisor Professor Alan Schumitzky. I also wish to thank Professor W. A. J. Luxemburg of the California Institute of Technology for his several very helpful suggestions.

**BIBLIOGRAPHY**


**DEPARTMENT OF MATHEMATICS, NORTHROP UNIVERSITY, INGLEWOOD, CALIFORNIA 90306**

**Current address**: 3115 Bagley Avenue, Los Angeles, California 90034