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COMPACT DYNAMICAL SYSTEMS[1]

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ABSTRACT. Compact flows on locally compact phase spaces are characterized in terms of the bilateral stability properties of the orbits and a stability theorem for cycles is formulated.

1. Introduction. A characterization of compact flows on dichotomic 2-manifolds in terms of the bilateral stability properties of the orbits was given by Knight in [5]. Our aim is to obtain such a characterization on a locally compact carrier. The Cycle Stability Theorem [4, p. 196] and Ura's Stability Theorem [2, Theorem 7.6] are fundamental tools used in [5]. The Cycle Stability Theorem applies only to certain 2-manifolds, consequently, we combine Ura's Alternatives [2, Theorem 9.5] with our Lemma 1 obtaining a cycle stability theorem for locally compact phase spaces.

Henceforth, we shall let (X, π) be a given flow on a Hausdorff phase space X. The periodic set and the critical set are denoted by P and S, respectively. The extension of a flow (X, π) to the one point compactification X^* is denoted by (X^*, π^*). The extension of a function F on X is denoted by F^* and the critical set of (X^*, π^*) is denoted by S^*. We denote the orbit through x, orbit closure of x, limit set of x, and prolongation of x by $C(x)$, $K(x)$, $L(x)$, and $D(x)$, respectively. By $A(M)$ we mean the region of bilateral attraction for a set M in X. The positive and negative versions of these concepts carry the appropriate $+$ or $-$ superscript. The reader is referred to [1] and [6] for standard definitions and notations used herein.

A flow is called *compact* whenever each of its orbits is compact. An orbit is compact if and only if it is periodic or critical [3, 3.09]. We say that a flow is of *characteristic* 0 *on a set* M if and only if $D(x) = K(x)$ for each $x \in M$. A flow is of *characteristic* 0 if it is of characteristic 0 on the phase space. The unilateral versions of this notion are defined similarly and carry the appropriate superscript.

2. Stability of cycles. We first obtain a lemma crucial to our cycle stability theorem. The lemma would be immediate if the periodic orbit conjecture held, however, an example of a flow on a compact 5-manifold with only periodic orbits which have unbounded periods was given by Sullivan [7].

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LEMMA 1. *If there are no critical points in a flow on a locally compact space, then the flow is of characteristic* 0.

PROOF. Let (X, π) be compact, X be locally compact, and the critical set S be empty. Then $X = P$ [3, 3.09]. We proceed by contradiction showing that $D(x) = C(x)$ for each x in P. For some x in P let y be a point of $D(x) - C(x)$. Select an open neighborhood V of $C(x)$ with compact closure contained in $X - C(y)$. Let $A = VR \cap (X - \overline{V})R$ and $D = X - A$. Then $C(x) \cup C(y) \subset D$, A is open, and $D \cap \partial V$ is compact. Define $t_z = \inf\{t \in R^+: zt \in V\}$ for $z \in A$. Evidently, $t_z < +\infty$. For any compact set $M \subset \partial V - D$ define $T_M = \sup\{t_z: z \in M\}$. For $z \in M$ there is a $zt \in V$ where $0 < t < t_z + 1$. The continuity of π yields $t_p \leqslant t < t_z + 1$ for each p in some neighborhood V_z of z. There is a finite subcover $\{V_{z_1}, \ldots, V_{z_k}\}$ of M. For each $p \in \bigcup \{V_{z_i}: 1 \leqslant i \leqslant k\}$ we have $t_p \leqslant \max\{t_{z_i} + 1: 1 \leqslant i \leqslant k\}$ so that $T_M < +\infty$. Next, let (x_i) be a net in V converging to x and let (r_i) be a net in R^+ such that $x_i r_i \to y$. There is a net $(x_{n_i}\tau_{n_i})$ in ∂V converging to a point z in ∂V where (x_{n_i}) is a subnet of (x_i). Choose τ_{n_i} to be the maximum value of R^+ in $[0, r_{n_i}]$ such that $x_{n_i}\tau_{n_i}$ is in ∂V. First, suppose that $z \in \partial V - D$. There is a compact neighborhood G of z disjoint from D. Let $M = G \cap \partial V$. Ultimately, $x_{n_i} r_{n_i} = (x_{n_i}\tau_{n_i})(r_{n_i} - \tau_{n_i}) \in M[0, T_M]$. Thus, $y \in M[0, T_M]$ and $C(y)$ meets $\overline{V}$ which is absurd. We must have z in D rather than in $\partial V - D$. For some $t > 0$ let G be a neighborhood of zt with compact closure such that $z \in X - \overline{G}$ and $C(x) \cup C(y) \subset X - \overline{G}$. If each compact subneighborhood of G contains a complete orbit, then there is a net (y_i) in $\overline{G}$ converging to zt such that $C(y_i) \subset \overline{G}$ for each i. But $z \in X - \overline{G}$ and we have $y_i(-t) \to z$ which is clearly impossible. Thus, choose $\overline{G}$ so that it contains no complete orbit. Now, either $C(z) \subset \overline{V}$ or $C(z) \subset X - V$ meeting both $X - \overline{V}$ and ∂V. First, we suppose that $C(z) \subset \overline{V}$ and let $V_0 = V - \overline{G}$. Then V_0 is a neighborhood of $C(x)$ with compact closure excluding $C(y)$. Using notation for V_0 similar to that used above for V, we have $z \in \partial V_0 - D_0$ so that $C(y)$ must meet ∂V_0 which is impossible. Hence, $C(z) \subset X - V$ but not ∂V. Let $T > 0$ be the fundamental period of z and let V_0 be the interior of $\overline{V}[0, T]$. Again we have a neighborhood V_0 of $C(x)$ with compact closure excluding $C(y)$. Furthermore, $\beta_{n_i} = \tau_{n_i} + T$ is the maximum value of $[0, r_{n_i}]$ such that $z_{n_i}\beta_{n_i}$ is in ∂V_0. We now have $z_{n_i}\beta_{n_i} \to zT = z$ in ∂V_0 where $C(z) \subset \overline{V}_0$. Already we have seen that this is absurd. Hence, we must conclude that $D(x) = C(x)$. The proof is complete.

According to Ura's Alternatives each periodic orbit $C(x)$ has one of the following properties. (i) $C(x)$ is asymptotically stable. (ii) $C(x)$ is negatively asymptotically stable. (iii) There exist points $y, z \notin C(x)$ such that $L^+(y) = L^-(z) = C(x)$. (iv) For every neighborhood V of $C(x)$ there is a complete orbit $C(y) \subset V - C(x)$.

In Lemma 1 we have shown that the orbits of P^0 satisfy only alternative (iv). Boundary orbits of P may have any of the properties. Those orbits having property (i) or (ii) are easily identified and are unilaterally stable. According to the Weak Attractor Theorem [**2**, Theorem 8.20] if $C(x)$ is a periodic bilateral attractor, then either $C(x)$ is a component of X or the smallest bilaterally stable set containing $C(x)$ is $D(x) = A(x)$. Moreover, any periodic orbit $C(x) \subset \partial P$ satisfying condition (iii) has $D(x) \neq C(x) \neq D^{\pm}(x)$ so that it cannot be bilaterally or unilaterally stable. The following theorem is a partial extension of the Cycle Stability Theorem to locally compact spaces.

THEOREM 2. *Let (X, π) be a flow on a locally compact space X. Then*

(i) *each orbit of P^0 is bilaterally stable,*

(ii) *a periodic boundary orbit $C(x)$ of P is not stable in any sense if there exist $y, z \notin C(x)$ such that $L^+(y) = L^-(z) = C(x)$, and*

(iii) *a periodic boundary orbit $C(x)$ of P is respectively stable, negatively stable, or bilaterally stable provided that the flow is of characteristic 0^+, 0^-, or 0 on $C(x)$.*

COROLLARY 2.1. *If (X, π) is a compact flow on a locally compact space X, then each orbit of P is bilaterally stable.*

COROLLARY 2.2. *On a locally compact phase space a compact flow is of characteristic 0 if and only if each critical point is bilaterally stable.*

PROPOSITION 3. *A compact flow on a locally compact space is of characteristic 0 if and only if it is of characteristic 0^+, 0^- or $0^{\pm}$.*

PROOF. In view of the fact that a compact flow on a locally compact space is of characteristic 0 if and only if its extended flow is of characteristic 0, Corollary 6.3 of [**6**] applies.

3. Compact flows. The following lemma is given in [**5**]. The proof is sufficient for locally compact carriers provided we use Theorem 2 in place of the Cycle Stability Theorem for dichotomic 2-manifolds.

LEMMA 4. *Let (X, π) be compact and X be locally compact. Each compact component of S is bilaterally stable.*

A flow with a compact 2-manifold minimal phase space is obtained by requiring $f(0, 0) > 0$ in the example of a flow on a torus described in [**6**]. Evidently, Theorem 4 of [**5**] is not valid for all 2-manifolds. Similar compact flows exist when $P \cup S$ is nonempty. A nest of concentric toral flows like the one above converging to a critical torus would be such a flow. For this reason we add an attraction criterion to Theorem 4 obtaining a characterization of compact flows for locally compact carriers.

THEOREM 5. *A flow on a locally compact space X is compact if and only if S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $L(X) = P \cup S$.*

PROOF. If the flow (X, π) is compact, then the three conditions follow from Theorem 2 and Lemma 4 applied to (X^*, π^*).

Conversely, we proceed by showing that P is open. For $y \in P$ select a compact invariant neighborhood V of $C(y)$ disjoint from S^*. Then $K(z)$ is compact for any $z \in V$ and $\varnothing \neq L(z) \subset P$. If $z \notin P$, select a point $q \in L(z)$ and a compact invariant neighborhood V_0 of $C(q)$ excluding z. This leads to the absurdity $K(z) \cap C(q) = \varnothing$. Hence, P is open.

Next, if $x \in X - P \cup S$, then $L^*(x) \subset S^*$ since P is open. Clearly this is impossible since a compact invariant neighborhood of S^* excluding x exists. Whence, (X, π) is compact.

COROLLARY 5.1. *A flow on a locally compact space X is compact if and only if each component of the boundary of S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $L(X) = P \cup S$.*

COROLLARY 5.2. *A closed flow on a locally compact space is compact if and only if S^* and each periodic orbit are bilaterally stable.*

THEOREM 6. *A flow on a locally compact space is compact if and only if S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $P \cup S$ is a global attractor.*

PROOF. A slight modification of the proof of Theorem 5 is required since the attraction condition yields $A^+(P \cup S) = L^+(X)$.

COROLLARY 6.1. *A flow on a connected locally compact space is compact if and only if S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $P \cup S$ is a closed attractor.*

PROOF. We need only consider sufficiency of the conditions. If $x \in A^+(P \cup S) - P \cup S$, then $L^*(x) \cap S^* \neq \varnothing$. As before this is not possible implying that $A^+(P \cup S) = P \cup S$ is open.

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