SKEW-PRODUCTS WITH SIMPLE APPROXIMATIONS

P. N. WHITMAN

Abstract. Conditions are given in order that the cartesian product of two measure-preserving invertible transformations admits an approximation. A class of skew-product transformations is defined and conditions are given for a member of this class to admit a simple approximation.

1. Preliminaries. Let \((X, \mathcal{F}, \mu)\) be a Lebesgue space; that is, a measure space isomorphic to the unit interval with Lebesgue measure. A measure-preserving invertible point transformation of \(X\) is called an automorphism of \((X, \mathcal{F}, \mu)\).

Let \(T: X \to X\) be an automorphism. The induced automorphism \(T_A: A \to A\) where \(A \subset \mathcal{F}\) is defined as follows:

\[ T_A x = T^k x, \quad x \in A \]

where \(k\) is the least positive integer such that \(T^k x \in A\).

Let \(Z\) denote the set of positive integers. Let \(f: X \to Z\) be an integrable function. The special automorphism over \(T\) built under the function \(f\) is defined as follows:

Put \(B(k, n) = \{(x, n): x \in X, f(x) = k), n, k \in Z\) and \(1 < n < k\). Put \(X(f) = \bigcup_{k \geq 1} \bigcup_{1 < n < k} B(k, n)\). Identify \(X\) with the set \(\bigcup_{k \geq 1} B(k, 1)\).

We may regard each set \(B(k, n), 1 < n < k,\) as a copy of \(B(k, 1)\). Consequently we may extend the measure \(\mu\) to \(X(f)\) and form a normalised measure \(\mu'\) on \(X(f)\) in the obvious way. We define \(T_f\), the special automorphism over \(T\), by

\[ T_f(x, n) = (x, n + 1) \quad \text{if} \quad 1 < n < f(x), \]

\[ T_f(x, f(x)) = (Tx, 1). \]

The following definitions are due to Katok and Stepin [4] and Chacon [1] respectively.

Definition 1. An automorphism \(T\) is said to admit a cyclic approximation by periodic transformations of the first kind (a.p.t.I) with speed \(f(n)\), where \(f(n)\) is a sequence of real numbers decreasing to zero, if there exists a sequence of partitions \(\{\xi(n)\}, \xi(n) = \{C_i(n): 1 \leq i \leq q(n)\}\) such that:

1. \(\xi(n) \to \varepsilon\);

2. \(\sum_{i=1}^{q(n)} \mu(TC_i(n) \Delta C_{i+1}(n)) < f(q(n))\), where \(C_{q(n)+1}(n)\) means \(C_1(n)\).

Received by the editors February 8, 1978.


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DEFINITION 2. An automorphism $T$ admits a simple approximation if there exists a sequence of partitions \{\xi(n)\}, $\xi(n) = \{C_i(n): 1 < i < q(n)\}$ such that:

1. $\xi(n) \rightarrow \varepsilon_X$;
2. $TC_i(n) = C_{i+1}(n), 1 < i < q(n) - 1$.

Let $R$ denote the positive reals.

We also require a slight adaption of Definition 1, as follows.

DEFINITION 3. An automorphism $T$ admits a cyclic a.p.t.I with speed $f(x)$, if $T$ admits a cyclic a.p.t.I with speed $f(n)$, where $f: R \rightarrow R$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

In [6] the following two results were shown.

THEOREM 1. Let $T: X \rightarrow X$ admit a cyclic a.p.t.I with speed $f(n) = o(1/n^2)$ with respect to a sequence of partitions $\xi(n)$, $\xi(n)$ having $q(n)$ elements. Let $A \in \mathcal{F}$ be approximated by sets $A(n) \leq \xi(n)$ with $A(n) \subseteq A$ such that $\mu(A \setminus A(n)) = o(1/q(n))$. Then $T_A$ admits a simple approximation.

THEOREM 2. Let $T: X \rightarrow X$ admit a cyclic a.p.t.I with speed $g(x) = o(1/x^k)$ with respect to a sequence of partitions $\{\xi(n)\}, \xi(n) = \{C_i(n): 1 < i < q(n)\}$. Let $f: X \rightarrow Z$ be integrable with $f(C_i(n)) = k_i(n) \in Z, 1 < i < q(n), n < 1$. Then $T_f$ admits a cyclic a.p.t.I with speed $G(n) = o(1/n^k)$.

2. Approximation of products. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be Lebesgue spaces. Let $S$ and $T$ be automorphisms of $X$ and $Y$ respectively. Let $\xi(n) = \{C_i(n): 1 < i < p(n)\}$ and $\zeta(n) = \{D_j(n): 1 < j < q(n)\}$ be partitions in $X$ and $Y$ respectively. Define $S_n$ and $T_n$ by

\[ S_n C_i(n) = C_{i+1}(n), 1 < i < p(n), \text{ where } C_{p(n)+1}(n) \]
\[ T_n D_j(n) = D_{j+1}(n), 1 < j < q(n), \text{ where } D_{q(n)+1}(n) \]

means $C_1(n)$, and

\[ T_n D_j(n) = D_{j+1}(n), 1 < j < q(n), \text{ where } D_{q(n)+1}(n) \]

means $D_1(n)$.

The lemma below is easily verified.

LEMMA 1. If $(p(n), q(n)) = 1$, then $S_n \times T_n$ maps the elements of $\xi(n) \times \zeta(n)$ cyclically; that is

\[ S_n \times T_n \left(C_i(n) \times D_j(n)\right) = C_{i+1}(n) \times D_{j+1}(n) \]

and $(S_n \times T_n)^k(C_i(n) \times D_j(n)) \neq C_i(n) \times D_j(n)$ for $k < p(n)q(n)$.

The theorem which follows gives the conditions for a product of automorphisms to admit a cyclic a.p.t.I with speed of the form $O(1/n^k)$.

THEOREM 3. Let $S: X \rightarrow X$ admit a cyclic a.p.t.I with speed $f(n)$ $O(1/n^{(r+1)k})$ where $r \geq 2, k \geq 1, r, k \in Z$, with respect to a sequence of partitions $\{\xi(n)\}, \xi(n) = \{C_i(n): 1 < i < p(n)\}$. Let $T: Y \rightarrow Y$ admit a cyclic a.p.t.I with speed $g(n) = O(1/n^{(r+1)k}),$ with respect to a sequence of partitions $\{\zeta(n)\}, \zeta(n) = \{D_j(n): 1 < j < q(n)\}$. Suppose that:
1. \((p(n), q(n)) = 1\);
2. \(p(n) < q(n)\);
3. \(p(n)^r > q(n)\).

Then \(S \times T\) admits a cyclic a.p.t.I with speed \(G(n) = 2\delta/n^k\), for some \(\delta > 0\).

**Proof.**

\[
\sum_{i=1}^{p(n)} \sum_{j=1}^{q(n)} \mu \times \nu(S \times T(C_i(n) \times D_j(n)) \Delta (C_{i+1}(n) \times D_{j+1}(n)))
\]

\[
< \sum_{i} \sum_{j} \mu(C_i(n)) \nu(TD_j(n) \Delta D_{j+1}(n)) + \nu(D_j(n)) \mu(SC_i(n) \Delta C_{i+1}(n))
\]

\[
< p(n) \mu(C_1(n)) \sum_{j} \nu(TD_j(n) \Delta D_{j+1}(n))
\]

\[
+ q(n) \nu(D_1(n)) \sum_{i} \mu(SC_i(n) \Delta C_{i+1}(n))
\]

\[
< f(p(n)) + g(q(n)).
\]

Since \(f(n) = O(1/n^{r+1})\), \(f(n) < \delta_1/n^{(r+1)k}\) for some \(\delta_1 > 0\). Similarly, \(g(n) < \delta_2/n^{(r+1)k}\) for some \(\delta_2 > 0\). Put \(\delta = \max\{\delta_1, \delta_2\}\) then

\[
f(p(n)) + g(q(n)) < \delta/p(n)^{(r+1)k} + \delta/q(n)^{(r+1)k} < 2\delta/p(n)^k q(n)^k,
\]

since by (3) \(p(n)^r > q(n)\).

Put \(G(n) = 2\delta/n^k\). By Lemma 1 and the above, \(\{\xi(n) \times \xi(n)\}\) is a sequence of partitions with respect to which \(S \times T\) admits a cyclic a.p.t.I with speed \(g(n)\).

In [1] it is shown that if an automorphism admits a cyclic a.p.t.I with speed \(\theta/n, \theta < 1\), then it has simple spectrum. Consequently if \(\delta < \frac{1}{2}\) and \(k > 1\), then \(S \times T\) will have simple spectrum. If \(f(n) = o(1/n^{(r+1)k})\) and \(g(n) = o(1/n^{(r+1)k})\) then it is easily seen that \(S \times T\) will have speed of approximation \(G(n) = 2\delta/n^k\) for any \(\delta > 0\).

If \(S\) and \(T\) and \(S \times T\) all have simple spectrum then \(S\) and \(T\) can have no common spectral type and consequently by a result of Hahn and Parry [3] \(S\) and \(T\) are disjoint.

If \(T\) has simple spectrum then \(T \times T\) has spectral multiplicity strictly greater than one. Consequently Theorem 3 shows that there are restrictions on the types of approximating partitions which exist for \(T\), when the speed of approximation is of the order of \(O(1/n^3)\).

In a similar way to Theorem 3 we can also show the following.

**Theorem 4.** Let \(S\): \(X \rightarrow X\) admit a cyclic a.p.t.I with speed \(f(n) = a/\log n, a > 0\), with respect to a sequence of partitions \(\{\xi(n)\}, \xi(n)\) having \(p(n)\) elements. Let \(T\): \(Y \rightarrow Y\) admit a cyclic a.p.t.I with speed \(g(n) = b/\log n, b > 0\), with
respect to a sequence of partitions \( \{\xi(n)\} \), \( \xi(n) \) having \( q(n) \) elements. Suppose \( p(n) \) and \( q(n) \) satisfy:

1. \( (p(n), q(n)) = 1 \);
2. \( p(n) < q(n) \);
3. \( \log q(n) < k \log p(n) \).

Then \( S \times T \) admits a cyclic a.p.t.I with speed \( g(n) = (k + 1)(a + b)/\log n \).

3. Skew-products with simple approximations. The class of skew-products we shall consider were discussed by Newton in [5] where formulae were given for calculating their entropy. Goodson has considered skew-products of a different type in [2]. He has given conditions for finite skew-products to admit a simple approximation.

Let \( S \) and \( T \) be automorphisms of \( X \) and \( Y \) respectively. Let \( f: X \to Z \) be integrable. The skew-products considered below are of the form

\[
\psi(x, y) = (Sx, T^{f(x)}y), \quad x \in X, y \in Y.
\]

Let \( \eta \) be the measure which assigns measure 1 to each point of \( Z \). Let \( V \) be the subset of \( X \times Y \times Z \) defined by \((x, y, i) \in V \) if \( i < f(x) \). So that \( V = V' \times Y \) where \( V' \) is the subset of \( X \times Z \) defined by \((x, i) \in V' \) if \( i < f(x) \). It is easily seen that

\[
\mu \times \nu \times \eta(V) = \mu \times \eta(V') = \int f(x) d\mu.
\]

We can consider \( V \) as a Lebesgue space with normalised measure \( \mu' \) defined by

\[
\mu'(A) = \mu \times \nu \times \eta(A) \cdot \left( \int f(x) d\mu \right)^{-1},
\]

where \( A \subset V \).

Define an automorphism \( \phi \) on \( V \) by

\[
\phi(x, y, i) = (x, Ty, i + 1) \quad \text{if } i < f(x),
\]

\[
= (Sx, Ty, 1) \quad \text{if } i = f(x).
\]

Then \( \phi = S_f \times T \). Furthermore it is clear that \( \psi \) is the automorphism induced by \( \phi \) on the set \( X \times Y \times \{1\} \).

**Definition 4.** Let \( \xi \) be a partition in \( X \) such that every element of \( \xi \) is contained in exactly one of the sets \( B(k, 1) \), defined in the first section, for some \( k \). Order the sets \( B(k, n), k \geq 1, 1 \leq n < k \), lexicographically. Then \( \xi' \) is the partition in \( X(f) \) consisting of the elements \( C \in \xi \), together with for each \( C \in \xi \), where \( C \subset B(k, 1) \), a copy of \( C \) in each of the sets \( B(k, n), 1 \leq n < k \). The ordering on \( \xi' \) is that inherited from the sets \( B(k, n) \).

We now give conditions in order that \( \psi \) should admit a simple approximation.

**Theorem 5.** Let \( S: X \to X \) admit a cyclic a.p.t.I with speed \( h(x) = O(1/x^{3(r+1)}) \), \( r \geq 2 \), \( r \in \mathbb{Z} \), with respect to a sequence of partitions \( \{\xi(n)\} \),
$\xi(n) = \{ C_i(n): 1 < i < s(n)\}$. Let $f: X \to Z$ be integrable with $f(C_i(n)) = k_i(n), 1 < i < s(n), n > 1$, and suppose that $\xi^f(n)$ has $p(n)$ elements. Let $T: Y \to Y$ admit a cyclic a.p.t.I with speed $g(n) = O(1/n^{3(r+1)})$ with respect to a sequence of partitions $\{\xi(n)\}, \xi(n) = \{ D_j(n): 1 < j < q(n)\}$. Suppose that:

1. $(p(n), q(n)) = 1$;
2. $p(n) < q(n)$;
3. $p(n)q(n) > q(n)$;
4. $s(n)^{r+1} \mu(X \setminus \bigcup_{i=1}^{s(n)} C_i(n)) \to 0$ as $n \to \infty$;
5. $q(n)^2 \mu(Y \setminus \bigcup_{j=1}^{q(n)} D_j(n)) \to 0$ as $n \to \infty$.

Then $\psi(x, y) = \psi_x y$ admits a simple approximation.

**Proof.** By Theorem 2, $S_f$ admits a cyclic a.p.t.I with speed $H(n) = O(1/n^{3(r+1)})$, with respect to the sequence of partitions $\xi^f(n)$.

By the remarks following Theorem 3, $\phi$ admits a cyclic a.p.t.I with speed $G(n) = O(1/n^2)$ with respect to the sequence of partitions $\{\xi^f(n) \times \xi(n)\}$. Now

$$p(n)q(n) \mu \left[ X \times Y \times \{1\} \setminus \sum_{i=1}^{s(n)} \sum_{j=1}^{q(n)} C_i(n) \times D_j(n) \times \{1\} \right]$$

$$< p(n)q(n) \mu \left[ X \setminus \sum_{i=1}^{s(n)} C_i(n) \right] + p(n)q(n) \nu \left[ Y \setminus \sum_{j=1}^{q(n)} D_j(n) \right]$$

$$< p(n)^{r+1} \mu \left[ X \setminus \bigcup_{i=1}^{s(n)} C_i(n) \right] q(n^2) \nu \left[ Y \setminus \bigcup_{j=1}^{q(n)} D_j(n) \right] \to 0 \text{ as } n \to \infty,$$

since $p(n) < s(n). (1 + 2f d\mu)$ for $n$ sufficiently large.

Hence $\phi_{x \times y \times \{1\}}$ admits a simple approximation by Theorem 1, which completes the proof.

It is fairly easy to manufacture examples of automorphisms $S$ and $T$ which satisfy the conditions of Theorem 5 by the stacking method. Using methods similar to those in [4], we can use continued-fraction theory to provide rotations of the unit interval, $S$ and $T$, which satisfy the conditions of Theorem 3. As a consequence of this, it is possible to give examples of skew-products, of the type discussed above, with interval exchange transformations in the base, and rotations in the fibres, which have simple spectrum, without using Theorem 5.

Chacon [1] has generalised the idea of cyclic a.p.t.I to that of approximation with multiplicity $N$. We remark that all the results shown above have straightforward generalisations to the 'multiplicity $N$ situation'. We then have the following generalisation of Theorem 5.

**THEOREM 6.** Let $S, f$ and $T$ be as in Theorem 5. Suppose that:

1. $g.c.d. (p(n), q(n)) = N$;
2. $p(n) < q(n)$;
3. \( p(n)^r > q(n) \);
4. \( s(n)^{r+1} \mu(X \setminus \bigcup_{n=1}^{s(n)} C_n) \to 0 \text{ as } n \to \infty \);
5. \( q(n)^{\nu} \nu(Y \setminus \bigcup_{n=1}^{q(n)} D_n) \to 0 \text{ as } n \to \infty \).

Then \( \psi(x, y) = (Sx, T^{f(x)}y) \) admits a simple approximation with multiplicity \( N \).

**BIBLIOGRAPHY**


**University of the Witwatersrand, Jan Smuts Avenue, Johannesburg, South Africa**