ON $L^1$-CONVERGENCE OF FOURIER SERIES
WITH QUASI-MONOTONE COEFFICIENTS

J. W. GARRETT, C. S. REES AND Č. V. STANOJEVIĆ

Abstract. For the class of Fourier series with quasi-monotone coefficients, it is proved that $\|s_n - a_n\| = o(1)$, $n \to \infty$, if and only if $a_n \lg n = o(1)$, $n \to \infty$. This generalizes a theorem for monotone coefficients and provides a new proof for a result due to Telyakovskii and Fomin.

The problem of $L^1$-convergence of the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

has been settled for various special classes of coefficients. W. H. Young [1] found that

$$a_n \lg n = o(1), \quad n \to \infty$$

is a necessary and sufficient condition for cosine series with convex ($\Delta^2a_n > 0$) coefficients, and A. N. Kolmogorov [2] extended that result to the cosine series with quasi-convex ($\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty$) coefficients. G. A. Fomin [3] showed that for cosine series with monotone coefficients (Y) is a sufficient condition and

$$a_n^2 / a_n \lg n = o(1), \quad n \to \infty$$

(•) is a necessary one. It is easy to see that $\Delta a_n > 0$ and (•) imply (Y). Hence, for cosine series with monotone coefficients such that (•) holds, the condition (Y) is necessary and sufficient for $L^1$-convergence. J. W. Garrett and Č. V. Stanojević [4] improved that result by showing that for trigonometric series with monotone coefficients such that

$$a_n \lg n = o(1), \quad n \to \infty, \quad \|s_n - f\| = o(1), \quad n \to \infty \iff f \in L^1$$

where $\| \cdot \|$ is the $L^1$-norm and

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

(For sine series an analogous result is due to E. Hille and J. D. Tamarkin [5].) Recently, S. A. Telyakovskii and G. A. Fomin [6] obtained a similar result for Fourier series with quasi-monotone coefficients. A sequence $\{a_n\}$, $a_n \to 0$, $n \to \infty$, is called quasi-monotone if for some $\alpha > 0$, the sequence $a_n / n^\alpha$ is...
monotonically decreasing (for $\alpha = 0$, the sequence is monotone).

The proof given in [6] involves certain results of S. M. Lozinskii [7] concerning summability and interpolation processes. In this paper, we shall show that the Telyakovskii-Fomin theorem follows from a generalization of Theorem 1 in [4]. Our proof depends only on an estimate of $\|s_n - \sigma_n\|$, where $\sigma_n$ is the Fejér sum.

**Theorem.** Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be a Fourier series with quasi-monotone coefficients. Then $\|s_n - \sigma_n\| = o(1)$, $n \to \infty$, if and only if $(a_n + b_n) \log n = o(1)$, $n \to \infty$.

**Proof.** We shall carry out the proof for the cosine series only, the proof for the sine series being essentially the same. For the “if part,” we need the following lemma.

**Lemma.** Let $(a_n)$ be a quasi-monotone sequence such that $a_n \log n = o(1), n \to \infty$. Then

$$\frac{1}{n} \sum_{k=1}^{n} |\Delta a_k| \log k = o(1), \quad n \to \infty.$$

**Proof.** From

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{n-1} \Delta (k a_k) \sum_{j=1}^{k} \frac{1}{j} + a_n \sum_{j=1}^{n} \frac{1}{j},$$

we obtain

$$\frac{1}{n} \sum_{k=1}^{n} k \Delta a_k \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{n} \sum_{k=1}^{n} a_k - a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n-1} \Delta a_{k+1} \sum_{j=1}^{k} \frac{1}{j}.$$

The quasi-monotonicity of the $(a_n)$ yields

$$|\Delta a_k| \leq \Delta a_k + 2 \alpha a_k / k$$

for some $\alpha > 0$. Hence,

$$\frac{1}{n} \sum_{k=1}^{n-1} k |\Delta a_k| \sum_{j=1}^{k} \frac{1}{j} \leq -a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n} a_k$$

$$+ \frac{2 \alpha}{n} \sum_{k=1}^{n-1} a_k \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n-1} \Delta a_{k+1} \sum_{j=1}^{k} \frac{1}{j}.$$

Each term on the right-hand side is $o(1)$ as $n \to \infty$. That completes the proof of the lemma.

Now
\[ \|s_n - \sigma_n\| = \frac{1}{n+1} \left\| \sum_{k=1}^{n} k\alpha_k \cos kx \right\| \]

\[ \leq \frac{1}{n+1} \left\| \sum_{k=1}^{n-1} k\Delta_k \left[ D_k(x) - \frac{1}{2} \right] \right\| \]

\[ + \frac{1}{n+1} \left\| \sum_{k=1}^{n-1} a_{k+1} \left[ D_k(x) - \frac{1}{2} \right] \right\| + a_n \left\| D_n(x) - \frac{1}{2} \right\| \]

where \( D_n(x) \) is the Dirichlet kernel. Or, since \( \|D_n(x) - 1/2\| = O(\log n) \), for some \( B > 0 \)

\[ B\|s_n - \sigma_n\| \leq \frac{1}{n+1} \sum_{k=1}^{n-1} k|\Delta_k|\log k + \frac{1}{n+1} \sum_{k=1}^{n-1} a_{k+1} \log k + a_n \log n. \]

From the lemma it follows that

\[ \|s_n - \sigma_n\| = o(1), \quad n \to \infty. \]

For the "only if" part, notice that

\[ \|s_n - \sigma_n\| + \|\sigma_n - f\| = \|s_n - f\| \geq C \sum_{k=1}^{n} \frac{a_{n+k}}{k} \]

where \( C \) is a positive constant. Since \( f \in L^1 \), we have that \( \|\sigma_n - f\| = o(1), \quad n \to \infty. \) Assume that \( \|s_n - \sigma_n\| = o(1), \quad n \to \infty. \) Then

\[ \sum_{k=1}^{n} \frac{a_{n+k}}{k} = o(1), \quad n \to \infty. \]

From the fact that the sequence \( \{a_n\} \) is quasi-monotone, we have

\[ \sum_{k=1}^{n} \frac{a_{n+k}}{k} > n^\alpha \sum_{k=1}^{n} \frac{1}{k} \frac{a_{n+k}}{(n+k)^{\alpha}} \geq \frac{n^\alpha a_{2n} \log n}{(2n)^{\alpha}} = \left( \frac{1}{2} \right)^\alpha a_{2n} \log n. \]

Finally, \( a_n \log n = o(1), \quad n \to \infty. \)

The proof of the Telyakovskii-Fomin theorem follows from the observation that if \( f \in L^1 \), then \( \|s_n - \sigma_n\| = o(1), \quad n \to \infty \iff \|s_n - f\| = o(1), \quad n \to \infty. \)

REFERENCES


Math tech, Inc., P. O. Box 2392, Princeton, New Jersey 08540

Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70122

Department of Mathematics, University of Missouri-Rolla, Rolla, Missouri 65401