PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 3, December 1978

HOMOCLINIC POINTS OF MAPPINGS OF THE INTERVAL

LOUIS BLOCK[1]

ABSTRACT. Let f be a continuous map of a closed interval I into itself. A point $x \in I$ is called a homoclinic point of f if there is a peridoic point p of f such that $x \neq p$, x is in the unstable manifold of p, and p is in the orbit of x under f^n, where n is the period of p. It is shown that f has a homoclinic point if and only if f has a periodic point whose period is not a power of 2. Furthermore, in this case, there is a subset X of I and a positive integer n, such that $f^n(X) = X$ and there is a topological semiconjugacy of f^n: $X \to X$ onto the full (one-sided) shift on two symbols.

1. Introduction. Let I denote a closed interval on the real line and $C^0(I, I)$ denote the set of continuous maps of I into itself. Let $f \in C^0(I, I)$. Bowen and Franks have shown in [3] that if f has a periodic point whose period is not a power of 2 then the topological entropy of f is positive. The converse to this theorem has been conjectured but is unproved. In trying to understand this situation, one may ask the following question: Is there some qualitative feature which occurs if and only if f has a periodic point whose period is not a power of 2? In this paper we give an answer to this question.

Before stating our answer, we define a homoclinic point for $f \in C^0(I, I)$. For a diffeomorphism of a smooth manifold a homoclinic point is defined to be a point which is in both the stable manifold and unstable manifold of some hyperbolic periodic point. Homoclinic points have been studied by Poincaré, Smale, and others in various aspects of differentiable dynamical systems. See [5] for more discussion and references.

To find a suitable definition of a homoclinic point for $f \in C^0(I, I)$ let us first suppose that f is differentiable and we have a hyperbolic fixed point p. A homoclinic point x must be in the unstable manifold of p, so p must be an expanding fixed point. Then p does not have a stable manifold in the usual sense. However, we may consider x to be in the stable manifold of p, in the sense that positive iterates of x approach p, if p is in the orbit of x. This, of course, cannot happen in the diffeomorphism case, but can happen if f is not one to one.

If p is not a fixed point of f, but a hyperbolic periodic point of period n, we may think of p as a hyperbolic fixed point of f^n and the previous discussion applies.

Presented to the Society, January 4, 1978; received by the editors August 12, 1977 and, in revised form, January 26, 1978.

AMS (MOS) subject classifications (1970). Primary 54H20.

[1]Partially supported by NSF Grant MCS 76-05822.

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In light of this duscussion we adopt the following definition. In this definition $W^u(p, f^n)$ denotes the unstable manifold, which we define in §2 (see also **[1]** and **[2]**).

DEFINITION. *Let $f \in C^0(I, I)$. A point $x \in I$ is a homoclinic point of f if there is a periodic point p of f such that the following hold.*

(1) *$x \neq p$.*

(2) *$x \in W^u(p, f^n)$ where n is the period of p.*

(3) *$f^{nm}(x) = p$ for some positive integer m.*

The main result of this paper is the following.

THEOREM A. *Let $f \in C^0(I, I)$. The following are equivalent.*

(i) *f has a periodic point whose period is not a power of* 2. (*Here, we include* $1 = 2^0$ *as a power of* 2.)

(ii) *f has a homoclinic point.*

(iii) *There are disjoint closed intervals J and K in I, and a positive integer n, such that $f^n(J) \supset J$, $f^n(J) \supset K$, $f^n(K) \supset J$, and $f^n(K) \supset K$.*

It is easy to see that if statement (iii) of Theorem A holds then there is a subset X of I, such that $f^n(X) = X$ and there is a topological semiconjugacy of $f^n: X \to X$ onto the full (one-sided) shift on two symbols. This implies, in particular, that the topological entropy of f is positive.

We will let Theorem A_1 denote the implication (i) $\Rightarrow$ (ii) of Theorem A, and Theorem A_2 denote the implication (ii) $\Rightarrow$ (iii). It is obvious that (iii) $\Rightarrow$ (i). After giving some definitions and preliminary lemmas in §2 we will prove Theorem A_1 in §3 and Theorem A_2 in §4.

The author would like to thank John Guckenheimer for a proof that (ii) $\Rightarrow$ (i) which is much shorter than the author's original proof of this fact. The proof of Theorem A_2 given here is just a modification of Guckenheimer's proof. The author would also like to thank Brian Marcus for helpful questions which lead to the inclusion of statement (iii) in Theorem A.

2. Preliminary definitions and results. Let $f \in C^0(I, I)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. We let f^0 denote the identity map of I.

Let $p \in I$. p is said to be a fixed point of f if $f(p) = p$. We say p is a periodic point of f, if p is a fixed point of f^n for some positive integer n. If p is a periodic point of f, the smallest positive integer n with $f^n(p) = p$ is called the period of p.

Now let p be a fixed point of f. We define the unstable manifold $W^u(p, f)$ and the one-sided unstable manifolds $W^u(p, f, +)$ and $W^u(p, f, -)$ as follows. Let $x \in W^u(p, f)$ if for every neighborhood V of p, $x \in f^n(V)$ for some positive integer n. Let $x \in W^u(p, f, +)$ if for every interval K with left endpoint p, $x \in f^n(K)$ for some positive integer n. Let $x \in W^u(p, f, -)$ if for every interval K with right endpoint p, $x \in f^n(K)$ for some positive integer n.

In the following lemma, we compile some basic properties of the unstable manifold. These properties follow easily from the definition.

LEMMA 1. *Let $f \in C^0(I, I)$ and let p be a fixed point of f.*

(1) $W^u(p, f)$, $W^u(p, f, +)$, and $W^u(p, f, -)$ are connected.

(2) $f(W^u(p, f)) \subset W^u(p, f)$, $f(W^u(p, f, +)) \subset W^u(p, f, +)$, and $f(W^u(p, f, -)) \subset W^u(p, f, -)$.

(3) $W^u(p, f, +) \cup W^u(p, f, -) = W^u(p, f)$.

Note that $W^u(p, f^n)$ is well defined when p is a fixed point of f^n. Also, if p is a fixed point of f^r and $n = kr$ (where n, k, and r are positive integers) then $W^u(p, f^n) \subset W^u(p, f^r)$. We use this fact in the next lemma.

LEMMA 2. *Let $f \in C^0(I, I)$. f has a homoclinic point if and only if for some positive integer n, there is a fixed point p of f^n and a point $z \in W^u(p, f^n)$ with $z \neq p$ and $f^n(z) = p$.*

PROOF. First, suppose that f has a homoclinic point x. Then for some periodic point p of f, $x \neq p$, $x \in W^u(p, f^n)$ where n is the period of p, and $f^{nm}(x) = p$ for some positive integer m. Let s be the smallest positive integer with $f^{ns}(x) = p$. Let $z = f^{n(s-1)}(x)$. Then p is a fixed point of f^n, $z \in W^u(p, f^n)$, $z \neq p$, and $f^n(z) = p$.

Now, suppose that for some positive integer n, there is a fixed point p of f^n and a point $z \in W^u(p, f^n)$ with $z \neq p$ and $f^n(z) = p$. Let r be the period of p. Then $n = kr$ for some positive integer k. We have $z \in W^u(p, f^r)$, $z \neq p$, and $f^{rk}(z) = p$. By definition, z is a homoclinic point of f. Q.E.D.

We conclude this section with the following elementary fixed point theorem.

LEMMA 3. *Let $f \in C^0(I, I)$. Let $K \subset I$ be a closed interval with $K \subset f(K)$. Then f has a fixed point in K.*

PROOF. For some points $x \in K$ and $y \in K$, $f(x)$ is the left endpoint of K and $f(y)$ is the right endpoint of K. Hence $f(x) \leqslant x$ and $f(y) \geqslant y$. By continuity, for some z in the closed interval joining x and y, $f(z) = z$. Q.E.D.

3. Proof of Theorem A_1. In following many of the proofs in the next two sections it would be helpful for the reader to make a diagram consisting of a line interval and points labeled in the correct order.

LEMMA 4. *Let $f \in C^0(I, I)$. Suppose p_1 and p_2 are fixed points of f with $p_1 < p_2$ and suppose that there are no fixed points of f in the interval (p_1, p_2). Then either $(p_1, p_2) \subset W^u(p_2, f, -)$ or $(p_1, p_2) \subset W^u(p_1, f, +)$.*

PROOF. Since f has no fixed points in (p_1, p_2), either $f(x) > x$ for all $x \in (p_1, p_2)$ or $f(x) < x$ for all $x \in (p_1, p_2)$. Without loss of generality we may assume that $f(x) > x$ for all $x \in (p_1, p_2)$.

Let $y \in (p_1, p_2)$. Let $[p_1, z]$ be any interval with $p_1 < z < y$. Note that the function $f(x) - x$ has a minimum value on the interval $[z, y]$, and this minimum value is positive. It follows that $y \in f^r([p_1, z])$ for some positive integer r. Hence $y \in W^u(p_1, f, +)$. Thus $(p_1, p_2) \subset W^u(p_1, f, +)$. Q.E.D.

THEOREM 5. *Let $f \in C^0(I, I)$ and suppose f has a periodic point of period* 3. *Then there is a fixed point p of f^2 and a point $z \in W^u(p, f^2)$ with $z \neq p$ and $f^2(z) = p$.*

PROOF. Let $\{p_1, p_2, p_3\}$ be a periodic orbit of f of period 3 with $p_1 < p_2 < p_3$. Either $f(p_1) = p_2$ or $f(p_3) = p_2$. We may assume without loss of generality that $f(p_1) = p_2$. Then $f(p_2) = p_3$ and $f(p_3) = p_1$. Hence $f([p_1, p_2]) \supset [p_2, p_3]$, $f([p_2, p_3]) \supset [p_1, p_2]$, and $f([p_2, p_3]) \supset [p_2, p_3]$.

Let $g = f^2$. Then $g([p_1, p_2]) \supset [p_1, p_2]$ and $g([p_2, p_3]) \supset [p_2, p_3]$. By Lemma 3, g has a fixed point in both $[p_1, p_2]$ and $[p_2, p_3]$.

Let $x_0 = \sup\{x \in [p_1, p_2]\colon g(x) = x\}$ and $y_0 = \inf\{x \in [p_2, p_3]\colon g(x) = x\}$. Then x_0 and y_0 are fixed points of g with $p_1 < x_0 < p_2 < y_0 < p_3$, and there are no fixed points of g in the interval (x_0, y_0). By Lemma 4, either $p_2 \in W^u(x_0, g, +)$ or $p_2 \in W^u(y_0, g, -)$.

We will assume that $p_2 \in W^u(y_0, g, -)$ (the proof is similar if $p_2 \in W^u(x_0, g, +)$). Since $g(W^u(y_0, g)) \subset W^u(y_0, g)$, $\{p_1, p_2, p_3\} \subset W^u(y_0, g)$. Hence $[p_1, p_3] \subset W^u(y_0, g)$.

Note that $g([p_1, p_2]) \supset [p_2, p_3]$. Hence for some point $z \in [p_1, p_2]$, $g(z) = y_0$. Now, y_0 is a fixed point of g, and $z \in W^u(y_0, g)$ with $z \neq y_0$ and $g(z) = y_0$. This proves Theorem 5. Q.E.D.

In proving Theorem A_1 we will use a theorem of Sarkovskii (see **[4]** or **[6]**) which says the following. Order the positive integers as follows: $1, 2, 4, 8, \ldots, 7 \cdot 8,\ 5 \cdot 8,\ 3 \cdot 8,\ \ldots, 7 \cdot 4,\ 5 \cdot 4,\ 3 \cdot 4, \ldots,\ 7 \cdot 2,\ 5 \cdot 2, 3 \cdot 2, \ldots, 7, 5, 3$. The theorem says if m is to the left of n, and $f \in C^0(I, I)$ has a periodic point of period n, then f has a periodic point of period m.

THEOREM A_1. *Let $f \in C^0(I, I)$. If f has a periodic point whose period is not a power of* 2, *then f has a homoclinic point.*

PROOF. Our hypothesis implies, by the theorem of Sarkovskii stated above, that for some positive integer r, f^r has a periodic point of period 3. Let $n = 2r$. By Theorem 5 (applied to f^r) there is a fixed point p of f^n and a point $z \in W^u(p, f^n)$ with $z \neq p$ and $f^n(z) = p$. By Lemma 2, f has a homoclinic point. Q.E.D.

4. Proof of Theorem A_2.

LEMMA 6. *Let $f \in C^0(I, I)$. Suppose there is a fixed point p of f and a point $y \neq p$ with $y \in W^u(p, f)$. Then for any neighborhood V of p there is a point $y_1 \in V \cap W^u(p, f)$ such that $f^r(y_1) = y$ for some positive integer r.*

PROOF. Suppose the conclusion is false. Then for some neighborhood V of $p, y \notin f^n(V \cap W^u(p, f))$ for every positive integer n. Since $y \in W^u(p, f)$, V

$\cap\ W^u(p, f)$ cannot be a neighborhood of p.

Without loss of generality, we may assume that $y > p$. Then p is the left endpoint of $W^u(p, f)$.

Now, V contains some open interval (a, b) where $a < p < b < y$. By continuity of f, $\exists c \in (a, p)$ such that $f(x) < b$ for all $x \in (c, p)$. Since $c \notin W^u(p, f)$, $\exists d \in (c, p)$ such that $c \notin f^n(d, p)$ for every positive integer n.

It follows that $y \notin f^n(d, b)$ for every positive integer n. This is a contradiction, because $y \in W^u(p, f)$ and (d, b) is a neighborhood of p. Q.E.D.

THEOREM A_2. *Let $f \in C^0(I, I)$ and suppose f has a homoclinic point. Then there are disjoint closed intervals J and K in I, and a positive integer n, such that $f^n(J) \supset J$, $f^n(J) \supset K$, $f^n(K) \supset K$, and $f^n(K) \supset J$.*

PROOF. By Lemma 2, for some positive integer s, there is a fixed point p of f^s and a point $y \in W^u(p, f^s)$ with $y \neq p$ and $f^s(y) = p$. Let $g = f^s$. Then p is a fixed point of g, and $y \in W^u(p, g)$ with $y \neq p$ and $g(y) = p$. Clearly, it suffices to show that the conclusion of the theorem holds for g (in place of f).

It follows by repeated application of Lemma 6, that there is a sequence (y_k) approaching p such that

(1) $y_0 = y$.

(2) For each positive integer k there is a positive integer r (depending on k) such that $g^r(y_k) = y_{k-1}$.

(3) $y_k \in W^u(p, g)$.

Without loss of generality we may assume that infinitely many y_k are to the right of p. Also, we may assume that $y > p$ (because if $y < p$, we could replace y by some $y_k > p$ in the rest of the proof).

Let x be any point of the sequence (y_k) in the interval (p, y). Then $g^m(x) = p$ for some positive integer m. Also, for some positive integer $n > m$, $\exists z \in (p, x)$ with $g^n(z) = y$ (i.e., z is some point in the sequence (y_k) with k sufficiently large). Since $n > m$, $g^n(x) = p$.

Now, $\exists z_1 \in (p, z)$ with $g^n(z_1) = x$ and $\exists z_2 \in (z, x)$ with $g^n(z_2) = x$. Let $J = [p, z_1]$ and $K = [z_2, x]$. Then $g^n(J) \supset J$, $g^n(J) \supset K$, $g^n(K) \supset K$, and $g^n(K) \supset J$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611