PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 3, December 1978

PROPERTIES OF S-CLOSED SPACES

DOUGLAS E. CAMERON[1]

ABSTRACT. It is shown that the S-closed spaces introduced by Travis Thompson using semiopen sets may be characterized as spaces where covers by regular closed sets have finite subcovers. S-closed is contagious and semiregular but is neither productive nor preserved by continuous surjections. Extremally disconnected QHC spaces are S-closed and maximal S-closed spaces are precisely the maximal QHC spaces which are extremally disconnected.

DEFINITION 1. A topological space (X, τ) is *quasi-H-closed* (denoted QHC) if every open cover has a finite proximate subcover (every open cover has a finite subfamily whose closures cover the space). In Hausdorff spaces, QHC is H-closed.

DEFINITION 2. A set A in a topological space (X, τ) is *semiopen* if $\operatorname{int}_\tau A \subset A \subset \operatorname{cl}_\tau \operatorname{int}_\tau A$ where $\operatorname{int}_\tau A$ denotes the interior of A with respect to the topology τ and $\operatorname{cl}_\tau A$ is the closure of A with respect to the topology τ.

DEFINITION 3. A topological space (X, τ) is *S-closed* if every semiopen cover has a finite proximate subcover.

It is obvious that every S-closed space is QHC but the converse is not true [7]. It has been shown that a space is QHC if and only if every cover of regular open sets has a finite proximate subcover. A similar result holds for S-closed spaces.

DEFINITION 4. A subset A of a topological space (X, τ) is *regular semiopen* if there is a regular open set U such that $U \subset A \subset \operatorname{cl}_\tau U$.

THEOREM 1. *A topological space (X, τ) is S-closed if and only if every cover of regular semiopen sets has a finite proximate subcover.*

PROOF. If the space is S-closed then the result follows.

If the space is not S-closed then there is a semiopen cover $\{A_\beta \mid \beta \in \mathfrak{B}\}$ which has no finite proximate subcover. Then $\{(\operatorname{int}_\tau \operatorname{cl}_\tau A_\beta) \cup A_\beta \mid \beta \in \mathfrak{B}\}$ is a regular semiopen cover which has no finite proximate subcover since $A_\beta \subset (\operatorname{int}_\tau \operatorname{cl}_\tau A_\beta) \cup A_\beta \subset \operatorname{cl}_\tau A_\beta$ for each $\beta \in \mathfrak{B}$. Q.E.D.

COROLLARY 1. *An extremally disconnected QHC space is S-closed.*

Received by the editors November 15, 1977 and, in revised form, April 10, 1978.

AMS (MOS) subject classifications (1970). Primary 54D20, 54D30, 54D99; Secondary 54G05.

Key words and phrases. S-closed, regular closed, extremally disconnected, semiopen, maximal topologies.

[1]The author wishes to thank the referee whose comments improved the exposition of this paper.

PROOF. This follows since in an extremally disconnected space, the regular open sets are clopen and thus the regular semiopen sets are clopen. Q.E.D.

Thus we have reverse implications to the results that Hausdorff or regular S-closed spaces are extremally disconnected. As a further means of characterizing S-closed spaces, we have the following.

THEOREM 2. *A topological space (X, τ) is S-closed if and only if every cover by regular closed sets has a finite subcover.*

Closely related to but not equivalent to Theorem 1 is the concept of semiregular. A topology τ on a set X is *semiregular* if τ has a base of regular open sets. The topology τ_S on X whose base is the regular open sets of τ is the *semiregularization* of τ. A topological property R is *semiregular* provided that a topological space (X, τ) has property R if and only if (X, τ_S) has property R. QHC and connectedness are semiregular.

If τ and τ' are two topologies on a set X, then τ' is an *expansion* of τ if $\tau \subset \tau'$. Two topologies are *ro-equivalent* if they have the same semiregularizations.

THEOREM 3. *An expansion τ' of τ is ro-equivalent to τ if and only if $\mathrm{cl}_\tau U = \mathrm{cl}_{\tau'} U$ for every $U \in \tau'$* [4].

THEOREM 4. *S-closed is semiregular.*

PROOF. This follows from Theorem 2 and the fact that τ and τ_S have the same regular closed sets. Q.E.D.

The following example shows that S-closed is not preserved by continuous surjections.

EXAMPLE 1. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be two copies of the natural numbers. Then $\beta\mathbf{N}_1$ and $\beta\mathbf{N}_2$ are S-closed [7] and extremally disconnected. The space X which is the free union of $\beta\mathbf{N}_1$ and $\beta\mathbf{N}_2$ is also S-closed and extremally disconnected. Let Y be the quotient of X by identifying a fixed point of $\beta\mathbf{N}_1 - \mathbf{N}_1$ with its counterpart in $\beta\mathbf{N}_2 - \mathbf{N}_2$. The quotient mapping is a continuous surjection, Y is Hausdorff but not extremally disconnected since the closures of $\mathbf{N}_1$ and $\mathbf{N}_2$ are not disjoint. Thus Y is not S-closed since Hausdorff S-closed spaces are extremally disconnected [7].

The preceding example also shows that quotient spaces of S-closed spaces are not necessarily S-closed. Also that a space which is the union of a finite number of regular closed subspaces which are S-closed in their relative topologies is not necessarily S-closed. However if the subspaces are clopen, then the result follows.

THEOREM 5. *(X, τ) is S-closed if X may be written as the union of a finite number of S-closed clopen subsets.*

Another relationship between an S-closed space and its regular closed subsets is seen in the fact that S-closed is regular closed hereditary [7].

THEOREM 6. *S-closed is contagious.* [*A property R is contagious if a space has the property whenever a dense subset has the property.*]

PROOF. Let A be an S-closed dense subset of (X, τ). If $\{0_\alpha | \alpha \in \mathfrak{A}\}$ is a semiopen cover of X, then $\{0_\alpha \cap A | \alpha \in \mathfrak{A}\}$ is a semiopen cover of A. Thus there is a finite proximate subcover $\{0_i \cap A | i = 1, 2, \ldots, m\}$ of A and thus $\{0_i | i = 1, 2, \ldots, m\}$ is a finite proximate subcover of X. Q.E.D.

EXAMPLE 2. Products of S-closed spaces need not be S-closed.

$\beta\mathbf{N}$ is S-closed but $\beta\mathbf{N} \times \beta\mathbf{N}$ is not extremally disconnected and since it is Hausdorff it is not S-closed.

THEOREM 7. *If $(\pi_{\mathfrak{A}} X_\alpha, \pi_{\mathfrak{A}} \tau_\alpha)$ is S-closed, then (X_α, τ_α) is S-closed for each $\alpha \in \mathfrak{A}$.*

PROOF. Let $\{\mathrm{cl}_{\tau_{\alpha_0}} U_\beta | \beta \in \mathfrak{B}\}$ be a regular closed cover of $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $\{\pi^{-1}\mathrm{cl}_{\tau_{\alpha_0}} U_\beta | \beta \in \mathfrak{B}\}$ is a regular closed cover of $\pi_{\mathfrak{A}} X_\alpha$ since $\mathrm{cl}_{\pi_{\mathfrak{A}}\tau_\alpha}(\pi_{\mathfrak{A}} A_\alpha) = \pi_{\mathfrak{A}} \mathrm{cl}_{\tau_\alpha} A_\alpha$. Thus there is a finite subcover $\{\pi^{-1}\mathrm{cl}_{\tau_{\alpha_0}} U_i | i = 1, 2, \ldots, n\}$ and thus $\{\mathrm{cl}_{\tau_{\alpha_0}} U_i | i = 1, 2, \ldots, n\}$ is a finite subcover of X_{α_0}. Q.E.D.

In the remainder of this paper we shall investigate maximal S-closed spaces.

DEFINITION 5. A topological space (X, τ) with property R is *maximal R* if no proper expansion of τ has property R.

If R is a semiregular topological property then maximal R spaces are submaximal (all dense subsets are open) [**2**]. Thus maximal S-closed spaces are submaximal.

THEOREM 8. *In a submaximal, extremally disconnected space (X, τ) all semiopen sets are open.*

PROOF. Let A be semiopen. Then $\mathrm{int}_\tau A \subset A \subset \mathrm{cl}_\tau A = \mathrm{cl}_\tau(\mathrm{int}_\tau A)$. Thus since subspaces of submaximal spaces are submaximal, A is a dense open subset of $\mathrm{cl}_\tau A$ and thus open since $\mathrm{cl}_\tau A$ is open. Q.E.D.

THEOREM 9. *A QHC subspace of a submaximal extremally disconnected space (X, τ) is S-closed.*

PROOF. Let A be a QHC subspace of (X, τ). Then $\mathrm{bd}_\tau A = \mathrm{cl}_\tau A - \mathrm{int}_\tau A$ is discrete since (X, τ) is submaximal. Thus $A - \mathrm{cl}_\tau \mathrm{int}_\tau A$ is a discrete clopen subset of A and thus QHC since QHC is regular closed hereditary. Therefore $A - \mathrm{cl}_\tau \mathrm{int}_\tau A$ is finite and thus S-closed. $A \cap \mathrm{cl}_\tau \mathrm{int}_\tau A$ is a dense subset of $\mathrm{cl}_\tau \mathrm{int}_\tau A$ and thus extremally disconnected since $\mathrm{cl}_\tau \mathrm{int}_\tau A$ is extremally disconnected as an open subset of X. By Theorem 8, every semiopen cover of $A \cap \mathrm{cl}_\tau \mathrm{int}_\tau A$ is an open cover and has a finite proximate subcover since $A \cap \mathrm{cl}_\tau \mathrm{int}_\tau A$ is QHC as a regular closed subset of the QHC space A. Thus A is S-closed by Theorem 5. Q.E.D.

THEOREM 10. *Extremally disconnected is semiregular.*

PROOF. If (X, τ) is extremally disconnected then for $U \in \tau_S$, $\mathrm{cl}_{\tau_S} U = \mathrm{cl}_\tau U \in \tau$ by Theorem 3, and thus $\mathrm{cl}_\tau U \in \tau_S$.

If (X, τ_S) is extremally disconnected then for $U \in \tau$, $\mathrm{cl}_\tau U = \mathrm{cl}_{\tau_S} U$. $\mathrm{int}_\tau \mathrm{cl}_\tau U \supset U$ and is regular open, so $\mathrm{cl}_\tau(\mathrm{int}_\tau \mathrm{cl}_\tau U) = \mathrm{cl}_\tau U \in \tau_S \subset \tau$. Thus (X, τ) is extremally disconnected. Q.E.D.

DEFINITION 6. A topological space (X, τ) with property R is *strongly* R if there exists an expansion which is maximal R.

COROLLARY 2. *An extremally disconnected maximal QHC space is maximal S-closed.*

COROLLARY 3. *A Hausdorff S-closed space is strongly S-closed.*

PROOF. An H-closed space has a finer ro-equivalent maximal H-closed topology [**4**]. Q.E.D.

THEOREM 11. *A maximal S-closed space is extremally disconnected.*

PROOF. Let (X, τ) be S-closed and B a regular closed subset, $B \notin \tau$. Claim $\tau(B) = \{U \cup (V \cap B) | U, V \in \tau\}$ is S-closed.

$$\mathrm{cl}_\tau(U \cup (V \cap B)) = \mathrm{cl}_\tau U \cup \mathrm{cl}_\tau(V \cap B)$$

and since B is closed, $\mathrm{cl}_\tau(V \cap B) \subset B$. If A is $\tau(B)$ semiopen then

$$U \cup (V \cap B) \subset A \subset \mathrm{cl}_{\tau(B)} U \cup \mathrm{cl}_\tau(V \cap B).$$

Let $x \in A - U \cup (V \cap B)$. If $x \in \mathrm{cl}_{\tau(B)} U$ then $x \in \mathrm{cl}_\tau U$ and if $x \in \mathrm{cl}_{\tau(B)}(V \cap B) = \mathrm{cl}_\tau(V \cap B)$ then $x \in \mathrm{cl}_\tau(V \cap \mathrm{int}_\tau B)$. Thus the semiopen sets of $\tau(B)$ are semiopen in τ and a $\tau(B)$ semiopen cover has a τ finite proximate subcover, say $\{U_i \cup (V_i \cap B) | i = 1, 2, \ldots, n\}$. Then $\{U_i | i = 1, 2, \ldots, n\}$ is a finite proximate cover of $X - B$ with respect to either τ or $\tau(B)$ since B is closed. Also since $\tau|B = \tau(B)|B$ and $B = \mathrm{cl}_\tau(\mathrm{int}_\tau B)$,

$$\{(U_i \cup (V_i \cap B)) \cap B = (U_i \cap B) \cup (V_i \cap B) | i = 1, 2, \ldots, n\}$$

is a finite proximate cover of B with respect to either τ or $\tau(B)$. Therefore $\{U_i \cup (V_i \cap B) | i = 1, 2, \ldots, n\}$ is a finite proximate $\tau(B)$ cover and $(X, \tau(B))$ is S-closed.

It follows that if (X, τ) is not extremally disconnected, then it is not maximal S-closed. Q.E.D.

THEOREM 12. *A maximal S-closed space (X, τ) is maximal QHC.*

PROOF. Let A be a subset of a maximal S-closed space such that A and $X - \mathrm{int}_\tau A$ are QHC. Since the space is extremally disconnected and submaximal, A and $X - \mathrm{int}_\tau A$ are S-closed. If A is not closed, then $(X, \tau(X - A))$ is QHC [**1**].

$$\tau|A = \tau(X - A)|A \text{ and } \tau|\mathrm{cl}_\tau(X - A) = \tau(X - A)|\mathrm{cl}_\tau(X - A).$$

Let $\{U_\alpha \cup C_\alpha | \alpha \in \mathfrak{A}, U_\alpha \in \tau(X - A)\}$ be a $\tau(X - A)$ semiopen cover of X.

$$A = [\mathrm{cl}_\tau(\mathrm{int}_\tau A) \cap A] \cup [A - \mathrm{cl}_\tau(\mathrm{int}_\tau A)].$$

Note: $\mathrm{int}_\tau A = \mathrm{int}_{\tau(X-A)} A$ and $\mathrm{cl}_{\tau(X-A)} \mathrm{int}_\tau A = \mathrm{cl}_\tau(\mathrm{int}_\tau A) \cap A$. $A - \mathrm{cl}_\tau(\mathrm{int}_\tau A)$ must be finite as in Theorem 9. $\mathrm{cl}_\tau(\mathrm{int}_\tau A) \cap A$ is a τ open subset of A since it is a τ semiopen set (Theorem 8) and thus it is $\tau(X - A)$ open. It is also $\tau(X - A)$ closed and thus $\tau(X - A)$ clopen and is S-closed as a regular closed subset of the S-closed space A.

Similarly $\mathrm{cl}_\tau(X - A) - \mathrm{cl}_\tau(\mathrm{int}_\tau(X - A))$ is finite and $\mathrm{cl}_\tau(\mathrm{int}_\tau(X - A))$ is a clopen subset of $\mathrm{cl}_\tau(X - A)$ and is S-closed and $\tau(X - A)$ clopen. Thus

$$X = \mathrm{cl}_\tau(\mathrm{int}_\tau(X - A)) \cup (A \cap \mathrm{cl}_\tau(\mathrm{int}_\tau A)) \cup B$$

where B is a finite set and clopen. Thus $(X, \tau(X - A))$ is S-closed which is a contradiction. A is closed and (X, τ) is maximal QHC [**1**]. Q.E.D.

Theorem 12 and Corollary 2 may be summarized in the following result.

THEOREM 13. *The maximal S-closed spaces are precisely the extremally disconnected maximal QHC spaces.*

COROLLARY 4. *In an extremally disconnected submaximal S-closed space (X, τ), the following are equivalent*:

(i) *(X, τ) is maximal S-closed,*

(ii) *(X, τ) is maximal QHC,*

(iii) *If A and $\mathrm{cl}_\tau(X - A)$ are QHC then A is closed,*

(iv) *if A and $\mathrm{cl}_\tau(X - A)$ are S-closed then A is closed.*

COROLLARY 5. *A maximal S-closed space is T_1. (Thus an S-closed space is strongly S-closed only if $\tau \vee \mathfrak{F}$ is S-closed where $\mathfrak{F}$ is the topology of finite complements.)*

PROOF. Maximal QHC spaces are T_1 [**1**]. Q.E.D.

EXAMPLE 3. A T_1 S-closed space which is not extremally disconnected.

Let $(\mathbf{N}^*, \tau^*)$ be the Katětov extension of the natural numbers [**6**]. This is the projective maximum among the H-closed extensions of a Hausdorff space and being a finer topology than the Stone-Čech compactification is maximal S-closed. Let (X, τ) be the free union of two copies of $(\mathbf{N}^*, \tau^*)$ and (X', τ') its one point compactification with ∞ the added point. Since (X, τ) is not locally compact, (X', τ') is not Hausdorff but is T_1. If C is a τ' regular closed set, then $C - \{\infty\}$ is τ regular closed. Thus any τ' cover of X' by regular closed sets may be made into a τ cover of X, a finite subcover selected and returned to a τ' cover by returning ∞ to each appropriate set and if necessary adding in a member of the original cover containing ∞.

EXAMPLE 4. An extremally disconnected submaximal S-closed space which is not strongly S-closed.

Let X be an infinite set, x_0 an arbitrarily chosen element. $U \in \tau$ if $U = \phi$ or $x_0 \in U$. $\tau \vee \mathfrak{F}$ is the discrete topology so this space is not strongly S-closed.

DEFINITION 7. The dispersion character of a space is the least cardinal number of a nonempty open set.

THEOREM 14. *If* (X, τ) *is a maximal S-closed space with infinite dispersion character, then any finer topology has an isolated point* [3].

REFERENCES

1. Douglas E. Cameron, *Maximal QHC-spaces*, Rocky Mountain J. Math. **7** (1977), 313–322.

2. _____, *A class of maximal topologies*, Pacific J. Math. **70** (1977), 101–104.

3. Murray Kirch, *On Hewitt's γ-maximal spaces*, J. Austral. Math. Soc. **14** (1972), 45–48.

4. J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected spaces*, Dissertationes Math. **66** (1969).

5. O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

6. Jack Porter and John Thomas, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. **138** (1969), 159–170.

7. Travis Thompson, *S-closed spaces*, Proc. Amer. Math. Soc. **60** (1976), 335–338.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AKRON, AKRON, OHIO 44325