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THE CONORMAL MODULE OF AN ALMOST COMPLETE INTERSECTION

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ABSTRACT. The conormal module of an ideal I in a commutative ring S is the S/I-module I/I^2. Assume S is a regular noetherian ring and I a prime ideal, which is locally everywhere a complete intersection or an almost complete intersection (i.e. needs one generator more than in the complete intersection case). In this situation necessary and sufficient conditions for I/I^2 being torsion free are given. Moreover the torsion of I/I^2 is expressed in terms of Kähler differentials of S/I.

1. Torsion freeness of the conormal module. Let S be a regular local ring, I an ideal of S and $R = S/I$. We say that I (or R) is a "complete intersection", if $\mu(I) = \operatorname{ht}(I)$, and that I (or R) is an "almost complete intersection", if $\mu(I) = \operatorname{ht}(I) + 1$. Here μ denotes the minimal number of generators and ht means "height".

I is a complete intersection iff the conormal module I/I^2 is a free R-module (see [3] or [9]). In this note we are interested in necessary and sufficient conditions for I/I^2 being torsion free, in case I is a prime ideal and an almost complete intersection. Observe that for a prime ideal I the R-module I/I^2 is torsion free iff I^2 is an I-primary ideal.

THEOREM 1. *Let S be a regular noetherian ring, I a prime ideal of S which is locally everywhere a complete intersection or an almost complete intersection. For $R = S/I$ let K_R be the canonical (dualizing) module of R, i.e. $K_R = \operatorname{Ext}_S^r(R, S)$, where $r = \operatorname{ht}(I)$. Then the following conditions are equivalent*:

(a) *I/I^2 is a torsion free R-module.*

(b) *K_R is a reflexive R-module.*

(c) *For all $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$ the local ring R_P is a complete intersection.*

From this we see, for example, that if under the assumptions of the theorem we have $\dim R = 1$ and R_M is an almost complete intersection for some maximal ideal M of R, then I^2 is not primary. Explicit examples in the polynomial ring $K[X_1, X_2, X_3]$ over a field K can easily be given. In fact, it was shown recently that for $I \in \operatorname{Spec}(K[X_1, X_2, X_3])$ the ideal I^2 is primary if

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and only if I is locally a complete intersection (J. Herzog, *Ein Cohen-Macaulay Kriterium mit Anwendungen auf den Konormalenmodul und den Differentialmodul*, Math. Z. (to appear).

Under the assumptions of the theorem condition (a) is "independent of the embedding", since condition (c) depends obviously only on R.

PROOF OF THEOREM 1.[1] It is enough to prove the local version of the theorem, so we shall assume that S is a regular local ring. We may also assume that R is an almost complete intersection, since the theorem is known for complete intersections. Matsuoka [7] has constructed an exact sequence

$$0 \to K_R \to R^{r+1} \to I/I^2 \to 0. \tag{1}$$

Moreover, Aoyama ([**1**, Lemma]) has shown the formula

$$\operatorname{depth}(K_{R_P}) = \operatorname{Min}\{2 + \operatorname{depth}(R_P), \dim R_P\} \tag{2}$$

for all $P \in \operatorname{Spec}(R)$.

Let C be the cokernel of $(I/I^2)^* \to (R^{r+1})^*$, where $*$ denotes the R-dual module. Then there is a linear map $\tau\colon K_R \to C^*$ such that the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_R & \longrightarrow & R^{r+1} & \longrightarrow & I/I^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^* & \longrightarrow & (R^{r+1})^{**} & \longrightarrow & (I/I^2)^{**} & & \end{array}$$

is commutative.

Suppose (a) is satisfied. Then $I/I^2 \to (I/I^2)^{**}$ is injective, since R is a domain, and therefore τ is an isomorphism. Since C^* is reflexive, being the dual of a finitely generated module over a noetherian domain, K_R is also reflexive. If K_R is reflexive, then so is K_{R_P} for all $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$. By [**4**, 7.29] R_p has to be Gorenstein. But R_p is an almost complete intersection or a complete intersection. By [**5**] only the second possibility can hold, hence (c) follows from (b).

Assume now that condition (c) of the theorem is satisfied. Then $\dim R_p \geq 2$, if R_p is an almost complete intersection; hence $\operatorname{depth}(K_{R_P}) \geq 2$ by (2) and $\operatorname{depth}(R_p \otimes_R I/I^2) \geq 1$ by (1). Thus P is not an associated prime of I/I^2. If R_P is a complete intersection, then $R_P \otimes_R I/I^2$ is even free. We conclude that I/I^2 is torsion free.

2. An exact sequence for the torsion of the conormal module. The torsion $T(I/I^2)$ of I/I^2 is related to the Kähler and Dedekind different of R over a suitable subring. In order to simplify we make the following assumptions: $S = k[X_1, \ldots, X_n]$ is a power series algebra over a perfect field k and $I \in \operatorname{Spec}(S)$.

[1] The author wishes to thank J. Herzog, T. Matsuoka and R. Waldi for comments leading to a generalization of the original theorem and a simplified proof.

In $R = S/I$ we write x_i for the image of X_i. If dim $R = d$, there is a power series algebra Q of d variables over k, such that $Q \subset R$, R is a Q-module of finite type and the quotient field L of R is separable algebraic over the quotient field K of Q.

After a change of variables, if necessary, we may assume that $Q = k[x_1, \ldots, x_d]$. We may identify Q with the subalgebra $k[X_1, \ldots, X_d]$ of S. Moreover we have an exact sequence

$$0 \to T(I/I^2) \to I/I^2 \to R \underset{S}{\otimes} D_Q(S) \to D_Q(R) \to 0, \qquad (3)$$

where D_Q is the Kähler differential module relative to Q. Suppose now I is an almost complete intersection of height $r = n - d$ and $\{F_1, \ldots, F_{r+1}\}$ a system of generators of I. We may assume that the mapping β: $R^{r+1} \to I/I^2$ in (1) sends the canonical basis element e_i of R^{r+1} to the image $\overline{F}_i$ of F_i in I/I^2 ($i = 1, \ldots, r + 1$). Combining (1) and (3) we get a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & K_R & = & K_R & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & D & \longrightarrow & R^{r+1} & \xrightarrow{\alpha} & R^r & \longrightarrow & D_Q(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\beta & & \downarrow & & \| & & \\
0 & \longrightarrow & T(I/I^2) & \longrightarrow & I/I^2 & \longrightarrow & R \otimes_S D_Q(S) & \longrightarrow & D_Q(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array} \qquad (4)$$

where α is given by the Jacobian matrix $J = (\partial F_i/\partial x_k)_{i=1,\ldots,r+1;k=d+1,\ldots,n}$ and $D = \ker(\alpha)$.

LEMMA 1. $D \cong \mathfrak{D}(R/Q)^{-1}$, *where* $\mathfrak{D}$ *is the Kähler different of* R *over* Q, *i.e. the ideal generated by all* $r \times r$ *minors of* J. *In particular, we have an exact sequence*

$$0 \to K_R \to \mathfrak{D}(R/Q)^{-1} \to T(I/I^2) \to 0.$$

PROOF. By tensoring the middle row of (4) with L we see that $D \otimes_R L \cong L$. By Cramer's rule $\ker(\alpha \otimes L) = L \cdot (\Delta_1 e_1 + \cdots + \Delta_{r+1} e_{r+1})$, where $\Delta_1, \ldots, \Delta_{r+1}$ are the $r \times r$ minors of J (with suitable signs). D can be identified with the set of all $\lambda \in L$ for which $\lambda\Delta_i \in R$ ($i = 1, \ldots, r + 1$), i.e. with $\mathfrak{D}(R/Q)^{-1}$.

3. Applications to differential forms. Under the assumptions as in the beginning of §2 we consider in the L-vector space $\Lambda^d(L \otimes_R D_k(R))$ of "meromorphic d-forms" the R-submodule

$$\Omega'_R := \mathfrak{D}(R/Q)^{-1} dx_1 \wedge \cdots \wedge dx_d.$$

Lemma 2. *Ω'_R does not depend on the choice of $Q \subset R$.*

Let $Q' = k[y_1, \dots, y_d]$ be another subalgebra of R having analogous properties as Q. In $\Lambda^d(L \otimes_R D_k(R))$ we have an equation

$$dx_1 \wedge \cdots \wedge dx_d = \delta dy_1 \wedge \cdots \wedge dy_d \qquad (\delta \in L \setminus \{0\})$$

and in $\Lambda^n(L \otimes_S D_k(S))$

$$dx_1 \wedge \cdots \wedge ds_d \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_r}$$
$$= \delta dy_1 \wedge \cdots \wedge dy_d \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_r},$$

if $F_{i_1}, \dots, F_{i_r}$ are taken from a set of generators $\{F_1, \dots, F_m\}$ of I. From this we can conclude that $\mathfrak{D}(R/Q) = \delta \mathfrak{D}(R/Q')$ and $\mathfrak{D}(R/Q)^{-1} dx_1 \wedge \cdots \wedge dx_d = \mathfrak{D}(R/Q')^{-1} dy_1 \wedge \cdots \wedge dy_d$.

Let $\mathfrak{C}(R/Q)$ be the Dedekind complementary module of R over Q, i.e. the set of all $\lambda \in L$ such that $\sigma_{L/K}(\lambda r) \in Q$ for all $r \in R$, where $\sigma_{L/K}\colon L \to K$ is the canonical trace. It is known that $\mathfrak{C}(R/Q) \subset \mathfrak{D}(R/Q)^{-1}$ and that $\mathfrak{C}(R/Q)$ is a canonical module of R. Moreover $\Omega_R := \mathfrak{C}(R/Q) \cdot dx_1 \wedge \cdots \wedge dx_d$ does not depend on the choice of Q (see **[6]**).

The R-modules Ω_R and Ω'_R represent two possibilities to define "regular d-forms for R". A third one is given by taking the image Ω''_R of $\Lambda^d D_k(R)$ in $\Lambda^d(L \otimes_R D_k(R))$.

If R is a regular local ring, then $\Omega_R = \Omega'_R = \Omega''_R$. For a complete intersection R still $\Omega_R = \Omega'_R$. The situation for almost complete intersections describes

Theorem 2. *Let R be an almost complete intersection. Then*

$$\Omega'_R / \Omega_R \cong T(I/I^2).$$

Hence the following conditions are equivalent:

(a) $\Omega_R = \Omega'_R$.

(b) R_P *is a complete intersection for all* $P \in \operatorname{Spec}(R)$, $\operatorname{ht}(P) = 1$.

(c) Ω_R *is reflexive.*

Proof. We shall use the construction of the exact sequence (1) given by Matsuoka [7]. By the "Primbasissatz" there is a minimal system of generators $\{F_1, \dots, F_{r+1}\}$ of I such that $\{F_1, \dots, F_r\}$ is an S-regular sequence and $IS_I = (F_1, \dots, F_r) \cdot S_I$.

Let $J := (F_1, \dots, F_r) \cdot S$. Then $K_R \cong J\colon I/J$ as R-module. There is a well-defined map $\gamma : J : I \to R^{r+1}$ given as follows: For $G \in J\colon I$ let $-GF_{r+1} = G_1F_1 + \cdots + G_rF_r$ ($G_i \in S$). Then γ maps G onto $\Sigma_{i=1}^r \overline{G}_i e_i + \overline{G} e_{r+1}$, where $\overline{G}$, $\overline{G}_i$ are the images of G, G_i in R. γ induces an injection $J\colon I/J \to R^{r+1}$ whose image is the kernel of $R^{r+1} \to I/I^2$.

Let $\Sigma = S/J$. We can choose $Q = k[X_1, \dots, X_d]$ such that Σ is a Q-module of finite type (and L separable algebraic over K, as before). Since

L is the residue field of S_I, we can conclude that

$$\Delta_{r+1} = \frac{\partial(F_1, \ldots, F_r)}{\partial(x_{d+1}, \ldots, x_n)} \neq 0.$$

We use diagram (4) with the sequence $0 \to K_R \to R^{r+1} \to I/I^2 \to 0$ as described above. With the notations as in the proof of Lemma 1 the mapping $D \to R^{r+1}$ identifies each $\lambda \in \mathfrak{D}(R/Q)^{-1}$ with $\lambda(\Delta_1 e_1 + \cdots + \Delta_{r+1}e_{r+1}) \in R^{r+1}$. This element is in $\ker(\beta)$ iff $\lambda\Delta_{r+1}$ is in the image of $J : I$ in R. In order to prove Theorem 2 it is therefore sufficient to show

LEMMA 3. *If Δ is the image of $J : I$ in R, then*

$$\Delta = \frac{\partial(F_1, \ldots, F_r)}{\partial(x_{d+1}, \ldots, x_n)} \cdot \mathfrak{E}(R/Q).$$

PROOF. Let $I' = J : I$. We have $J = I \cap I'$ and I' has only associated primes $P_1, \ldots, P_s$ of height r and different from I. If $\bar{I}$, $\overline{I'}$ and $\bar{P}_i$ ($i = 1, \ldots, s$) denote the images in Σ, then $\bar{I} \cap \overline{I'} = (0)$, $\overline{I'} = \operatorname{Ann}_\Sigma(\bar{I})$ and $\Sigma_{\bar{I}} = L$. For the full ring of quotients of Σ we have

$$Q(\Sigma) = K \underset{Q}{\otimes} \Sigma = L \times \Sigma_{\bar{P}_1} \times \cdots \times \Sigma_{\bar{P}_s}. \tag{5}$$

The image of $\overline{I'}$ in $Q(\Sigma)$ is $\Delta \times (0) \times \cdots \times (0)$.

In the commutative diagram of canonical homomorphisms

$$\begin{array}{ccc} \operatorname{Hom}_Q(R, Q) & \xrightarrow{\alpha} & \operatorname{Hom}_K(L, K) \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ \operatorname{Hom}_Q(\Sigma, Q) & \longrightarrow & \operatorname{Hom}_K(K \otimes_Q \Sigma, K) \end{array}$$

all mappings are injective. Let $\sigma_{L/K}: L \to K$, $\sigma_{\Sigma/Q}: \Sigma \to Q$ and $\sigma: K \otimes_Q \Sigma \to K$ be the canonical traces. We have $\operatorname{Hom}_K(L, K) = L\sigma_{L/K}$ and $\operatorname{im}(\alpha) = \mathfrak{E}(R/Q)\sigma_{L/K}$. Moreover, since Σ/Q is a complete intersection, $\operatorname{Hom}_Q(\Sigma, Q) = \Sigma \cdot \eta$ with a trace map $\eta: \Sigma \to Q$, which by Scheja-Storch [**8**, 4.2], can be chosen in such a way that

$$\sigma_{\Sigma/Q} = \frac{\partial(F_1, \ldots, F_r)}{\partial\left(\bar{X}_{d+1}, \ldots, \bar{X}_n\right)} \cdot \eta,$$

where $\partial(F_1, \ldots, F_r)/\partial(\bar{X}_{d+1}, \ldots, \bar{X}_n)$ denotes the image of the Jacobian determinant in Σ.

We have $\operatorname{im}(\beta) = \overline{I'}\eta$, since for $s \in \Sigma$ the map $s\eta$ factors through R iff $s \in \bar{I}'$. From (5) we get a decomposition

$$\begin{aligned} \operatorname{Hom}_K\Big(K \underset{Q}{\otimes} \Sigma, K\Big) &= \left(L \times \Sigma_{\bar{P}_1} \times \cdots \times \Sigma_{\bar{P}_s}\right) \cdot \eta \\ &= \operatorname{Hom}_K(L, K) \times \operatorname{Hom}_K(\Sigma_{\bar{P}_1}, K) \times \cdots \times \operatorname{Hom}_K(\Sigma_{\bar{P}_s}, K), \end{aligned}$$

and $\mathrm{Hom}_K(L, K) \to \mathrm{Hom}_K(K \otimes_Q \Sigma, K)$ is the canonical injection onto the first factor.

The image of $\overline{I'}\eta$ in $\mathrm{Hom}_K(K \otimes_Q \Sigma, K)$ is

$$(\Delta \times (0) \times \cdots \times (0)) \cdot \eta = \left[\Delta_{r+1}^{-1} \cdot (\Delta \times (0) \times \cdots \times (0))\right] \cdot \sigma$$
$$= \left(\Delta_{r+1}^{-1} \cdot \Delta \cdot \sigma_{L/K}\right) \times (0) \times \cdots \times (0).$$

We obtain $\mathfrak{C}(R/Q) \cdot \sigma_{L/K} = \Delta_{r+1}^{-1} \cdot \Delta \cdot \sigma_{L/K}$, which proves the claim.

If R is an almost complete intersection of dimension 1 the length of $T(I/I^2)$ is related to the length of the torsion $T(D_k(R))$ of the differential module.

Consider the exact sequence

$$0 \to T(D_k(R)) \to D_k(R) \to \Omega_R \to C \to 0,$$

where C is the cokernel of the canonical map $D_k(R) \to \Omega_R$.

If $Q \subset R$, $Q = k[x]$ is chosen as above, then Rdx is a free submodule of $D_k(R)$ and hence we have also exact sequences

$$0 \to T(D_k(R)) \to D_k(R)/Rdx \to \Omega_R/Rdx \to C \to 0$$

and

$$0 \to \Omega_R/Rdx \to \Omega'_R/Rdx \to T(I/I^2) \to 0.$$

This gives us the length-formula

$$l(C) = l\big(T(D_k(R))\big) - l\big(T(I/I^2)\big) + l(\Omega'_R/Rdx) - l(D_k(R)/Rdx).$$

By Berger [2, Satz 2], $l(D_k(R)/Rdx) = l(\mathfrak{D}(R/Q)^{-1}/R) = l(\Omega'_R/Rdx)$, therefore

$$l(C) = l\big(T(D_k(R))\big) - l\big(T(I/I^2)\big).$$

Since $\dim R = 1$ we have $T(I/I^2) \neq 0$, hence

$$0 < l\big(T(I/I^2)\big) \leqslant l\big(T(D_k(R))\big)$$

and

$$l(C) < l\big(T(D_k(R))\big).$$

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