PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 1, January 1979

A NONLINEAR THEOREM OF ERGODIC TYPE.II

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ABSTRACT. A theorem is proved concerning the relationship between a certain collection of mappings and a fixed point for that collection of mappings. The conditions on the mappings are very similar to ones in a paper by W. F. Eberlein.

THEOREM. *Let B be a uniformly convex Banach space with a weak duality mapping* [2] *and C a closed convex subset of B. Suppose $G = \{S_m\}_{m=1}^{\infty}$ is a collection of self-mappings of C such that $I - S_m$ is demiclosed $\forall S_m \in G$. Suppose further that there is a collection of nonexpansive self-mappings of C, $\{C_n\}_{n=1}^{\infty}$ satisfying*

(a) $C_n(x) \in \{S_m(x)\colon S_m \in G\}^-$ *for all* $x \in C$,

(b) *there exist* x_0 *such that*

$$C_nS_m(x_0) - C_n(x_0) \to 0 \quad \text{as } n \to \infty,$$
$$S_mC_n(x_0) - C_n(x_0) \to 0 \quad \text{as } n \to \infty,$$

(c) *suppose there exists a subsequence* $\{C_{n_j}(x_0)\}$ *of* $\{C_n(x_0)\}_{n=1}^{\infty}$ *such that*

$$C_{n_j}(x_0) \rightharpoonup y \in C \quad [\text{since } C \text{ is weakly closed}]$$

then

(1) $S_m(y) = y$ *for all* $S_m \in G$,

(2) $C_n(x_0) \rightharpoonup y$.

LEMMA 1. *Under the above conditions the following is true*: $C_n(S(x_0)) - C_n(x_0) \to 0$ *as* $n \to \infty$ *where* $S(x_0)$ *is any element of* $\{S_m(x_0)\colon S_m \in G\}^-$.

PROOF. Clearly we need only check limit points of $\{S_m(x_0)\colon S_m \in G\}$. So let $S(x_0)$ be a limit point of $\{S_m(x_0)\colon S_m \in G\}$, and let $S_{m_j}(x_0) \to S(x_0)$. Given $\varepsilon > 0$ there exists m_{j_1} such that $\|S_{m_j}(x_0) - S(x_0)\| < \varepsilon/2$ for all $m_j \geqslant m_{j_1}$. Now consider

$$\begin{aligned}
&\|C_n(S(x_0)) - C_n(x_0)\| \\
&\quad = \|C_n(S(x_0)) - C_n(S_{m_j}(x_0)) + C_n(S_{m_j}(x_0)) - C_n(x_0)\| \\
&\quad \leqslant \|C_n(S(x_0)) - C_n(S_{m_j}(x_0))\| + \|C_nS_{m_j}(x_0) - C_n(x_0)\|
\end{aligned}$$

Received by the editors September 26, 1977.
AMS (MOS) subject classifications (1970). Primary 47H10; Secondary 47H99.
0002-9939/79/0000-0008/$0.1.75

which if we pick $m_{j_2} > m_{j_1}$ we have that

$$\|C_n(S(x_0)) - C_n(x_0)\|$$
$$\leqslant \|C_n(S(x_0)) - C_n(S_{m_{j_2}}(x_0))\| + \|C_n(S_{m_{j_2}}(x_0)) - C_n(x_0)\|$$
$$- \|S(x_0) - S_{m_{j_2}}(x_0)\| + \|C_n(S_{m_{j_2}}(x_0)) - C_n(x_0)\|$$

[by nonexpansiveness of the C_n's] $\leqslant \varepsilon/2 + \|C_n(S_{m_{j_2}}(x_0)) - C_n(x_0)\|$ for which there exists n_0 such that for all $n \geqslant n_0 \leqslant \varepsilon/2 + \varepsilon/2$ (by condition (b)). So for all $n \geqslant n_0$ we have $\|C_n(S(x_0)) - C_n(x_0)\| < \varepsilon \Rightarrow \|C_n(S(x_0)) - C_n(x_0)\| \to 0$ as $n \to \infty$.

We will now show $S_m(y) = y$ for all $S_m \in G$. We have that $C_{n_j}(x_0) \rightharpoonup y$, from now on C_{n_j} will be reindexed by L, i.e. $C_L(x_0) \rightharpoonup y$. By condition (c) $(I - S_m)C_n(x_0) \to 0 \Rightarrow (I - S_m)C_L(x_0) \to 0$ so by the demiclosedness of $I - S_m$ (for all $S_m \in G$) we have $(I - S_m)y = 0$ or $y = S_m(y)$ for all $S_m \in G$. Also since $C_n(y) \in \{S_m(y)\colon S_m \in G\}^- = \{y\}^- = \{y\}$ we have $C_n(y) = y$ for all n.

The set $\{C_n(x_0)\}$ is a bounded set because $\|C_n(x_0) - y\| = \|C_n(x_0) - C_n(y)\| \leqslant \|x_0 - y\| \Rightarrow \|C_n(x_0)\| \leqslant \|x_0 - y\| + \|y\|$ for all n. Consider the subset $\{C_n(x_0)\} \setminus \{C_L(x_0)\} \subseteq \{C_n(x_0)\}$ this is clearly bounded which implies, since uniformly convex Banach spaces are reflexive, that there exists a weakly convergent subsequence, call it $C_s(x_0) \rightharpoonup y_1$. We will show $y = y_1$.

Now by nonexpansiveness of the C_n's we have $\|C_L(C_s(x_0)) - y\| = \|C_L(C_s(x_0)) - C_L(y)\| \leqslant \|C_s(x_0) - y\|$.

We know that $C_L(C_s(x_0)) - C_L(x_0) \to 0$ as $L \to \infty$ by assumption (2), Lemma 1, and the fact that $C_s(x_0) \in \{S_m(x_0)\colon S_m \in G\}^-$. So there exists L_0 such that for all $L \geqslant L_0 \|C_L(C_s(x_0)) - C_L(x_0)\| < \varepsilon$, given $\varepsilon < 0$. Now $|\, \|C_L C_s(x_0) - y\| - \|C_L(x_0) - y\|\, | \leqslant \|C_L(C_s(x_0)) - C_L(x_0)\| < \varepsilon$ for all $L \geqslant L_0 \Rightarrow$ that $\|C_L(x_0) - y\| \leqslant \|C_L(C_s(x_0)) - y\| + \varepsilon \leqslant \|C_s(x_0) - y\| + \varepsilon$ for all $L \geqslant L \Rightarrow \underline{\mathrm{Lim}}\{\|C_L(x_0) - y\|\} \leqslant \|C_s(x_0) - y\|$. This holds for all $C_s(x_0) \Rightarrow \underline{\mathrm{Lim}}\{\|C_L(x_0) - y\|\} \leqslant \underline{\mathrm{Lim}}\{\|C_S(x_0) - y\|\}$. Now by interchanging the roles of C_L and C_S we get $\underline{\mathrm{Lim}}\{\|C_S(x_0) - y\|\} \leqslant \underline{\mathrm{Lim}}\{\|C_L(x_0) - y\|\}$ which $\Rightarrow$

$$\underline{\mathrm{Lim}}\{\|C_S(x_0) - y\|\} = \underline{\mathrm{Lim}}\{\|C_L(x_0) - y\|\}.$$

Let $d = \underline{\mathrm{Lim}}\{\|C_S(x_0) - y\|\}$ then extract a subsequence such that $d = \mathrm{Lim}\|C_T(x_0) - y\|$, and $d = \mathrm{Lim}\|C_U(x_0) - y\|$ where $C_T(x_0)$ is a subsequence of $C_S(x_0)$ and $C_U(x_0)$ is a subsequence of $C_L(x_0)$. Now replace in the previous argument $C_L(x_0)$ by $C_U(x_0)$, $C_S(x_0)$ by $C_T(x_0)$ and y by y_1 [using the fact that since $C_S(x_0) \rightharpoonup y_1 \Rightarrow C_n(y_1) = y_1$, same proof as before with y replaced by y_1]. The result of this is that $\underline{\mathrm{Lim}}\{\|C_U(x_0) - y_1\|\} = \underline{\mathrm{Lim}}\{\|C_T(x_0) - y_1\|\}$, so as before extract subsequences such that

$$d_1 = \underline{\mathrm{Lim}}\{\|C_U(x_0) - y_1\|\} = \mathrm{Lim}\|C_V(x_0) - y_1\| = \mathrm{Lim}\|C_W(x_0) - y_1\|$$

where $C_V(x_0)$ is a subsequence of $C_U(x_0)$ and $C_W(x_0)$ is a subsequence of

$C_T(x_0)$. We now have that

$$d = \operatorname{Lim}\|C_V(x_0) - y\| = \operatorname{Lim}\|C_W(x_0) - y\|,$$

$$d_1 = \operatorname{Lim}\|C_V(x_0) - y_1\| = \operatorname{Lim}\|C_W(x_0) - y_1\| \quad \text{and}$$

$$C_V(x_0) \rightharpoonup y, \qquad C_W(x_0) \rightharpoonup y_1.$$

Now using Lemma 3 of [**2**] we have that, since $C_V(x_0) \rightharpoonup y$, $\operatorname{Lim}\|C_V(x_0) - y\| \leqslant \operatorname{Lim}\|C_V(x_0) - y_1\|$ i.e. $d \leqslant d_1$, but since $C_W(x_0) \rightharpoonup y_1$ this implies $\operatorname{Lim}\|C_W(x_0) - y_1\| \leqslant \operatorname{Lim}\|C_W(x_0) - y\|$ i.e. $d \leqslant d_1$ so that $d = d_1$, which by Lemma 3 $y = y_1$.

REMARK 1. This is a generalization of Opial's Theorem 2 in [**2**], with $S_m = T^m$ and $C_n = T^n$. Opial in [**2**] showed that if T was nonexpansive then $I - T^m$ was demiclosed in a uniformly convex Banach space with a weakly continuous duality mapping, so $\{S_m\}$ satisfying the conditions imposed on them. Also condition (1) is trivially satisfied, and the asymptotic regularity of Γ implies condition (2). Since T has a fixed point by assumption this implies there exists a weakly convergent subsequence, for each $x \in C_1$ of $\{C_n(x)\}_{n=1}^{\infty}$, i.e. $C_{n_S}(x) \rightharpoonup y_x$. So our theorem says that

(1) $T^n(y_x) = y_x \ \forall n$ and

(2) $C_n(x) \rightharpoonup y_x$.

Also since Opial's Theorem 2 implies Theorem 3 our theorem implies Theorem 3, which is a partial affirmative answer to H. Schaefer's conjecture. Unfortunately it adds no new information on the conjecture.

REMARK 2. The conditions of this theorem are very similar to the ones of the mean ergodic theorem of Eberlein [**1**].

REFERENCES

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