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A MULTIPLIER THEOREM FOR $H^1(R^n)$

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ABSTRACT. We prove the following theorem.

THEOREM. *Let m be a nonnegative measurable function on* $[0, \infty)$. *For $n > 2$, the two conditions below are equivalent*:

(a) $\int_{R^n}|\hat{f}(x)|m(|x|)\,dx < \infty$ *for each* $f \in H^1(R^n)$,

(b) $\sup\{2^{(n-1)k}\int_{2^k}^{2^{k+1}} m(r)\,dr: -\infty < k < \infty\} < \infty$.

1. Introduction.

Notation. Let x stand for a point in R^n. For $x, y \in R^n$, $x \cdot y$ will denote the usual inner product, and $|x|$ will stand for $(x \cdot x)^{1/2}$. Lebesgue measure on R^n will be denoted dx. For $f \in L^1(R^n)$, $\hat{f}(x)$ will denote the Fourier transform of f: $\hat{f}(x) = \int_{R^n} f(y)e^{-2\pi i x \cdot y}\,dy$.

Consider the following two conditions which a nonnegative measurable function m on R^n might satisfy:

(a′) $\int_{R^n}|\hat{f}(x)|m(x)\,dx < \infty$ for each $f \in H^1(R^n)$,

(b′) $\sup\{\int_{2^k \leq |x| \leq 2^{k+1}} m(x)\,dx: -\infty < k < \infty\} < \infty$.

In Theorem 3 of **[3]**, R. Johnson proved that (a′) implies (b′). On the other hand, implicit in **[1]** is the proof that (a′) is implied by the following strengthened version of (b′):

$$\sup\left\{2^{nk(p-1)}\int_{2^k \leq |x| \leq 2^{k+1}} \left[m(x)\right]^p dx: -\infty < k < \infty\right\} < \infty \quad \text{for some } p > 1.$$

The purpose of this paper is to show that for $n \geq 2$ and for radial functions $m(x)$, (b′) implies (a′). Thus we will establish the implication (b) implies (a) of the following theorem.

THEOREM. *Let m be a nonnegative measurable function on* $[0, \infty)$. *For $n \geq 2$, the two conditions below are equivalent*:

(a) $\int_{R^n}|\hat{f}(x)|m(|x|) < \infty$ *for each* $f \in H^1(R^n)$,

(b) $\sup\{2^{(n-1)k}\int_{2^k}^{2^{k+1}} m(r)\,dr: -\infty < k < \infty\} < \infty$.

For the sake of completeness, we will include in §3 a proof of the fact that (b′) does not in general imply (a′).

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2. Proof that (b) implies (a). Let Σ stand for the unit sphere in R^n and let $d\sigma(x')$ denote Lebesgue measure on Σ. Since

$$\int_{R^n} |\hat{f}(x)| m(|x|)\, dx = \int_0^\infty m(r) \int_\Sigma |\hat{f}(rx')| d\sigma(x') r^{n-1}\, dr,$$

it is enough to show that

$$\sum_{k=-\infty}^{\infty} \sup\left\{ \int_\Sigma |\hat{f}(rx')| d\sigma(x') : 2^k \leqslant r \leqslant 2^{k+1} \right\} \tag{1}$$

is finite for all $f \in H^1(R^n)$. This will be done by making use of the "atomic" theory of H^1 as set forth in [2]. (The reader may consult [2] for the definitions of unfamiliar terms.) In the following, C will denote a positive constant independent of f and g but which may change from line to line.

It is necessary to show only that the sum (1) is bounded by some C for all $(1, \infty)$-atoms f. This will be done by establishing the following lemmas.

LEMMA 1. *There exists a positive constant C such that if g is any* $(1, 2)$-*atom centered at the origin in R^n, then*

$$\sum_{k=-\infty}^{\infty} \sup_{2^k \leqslant r \leqslant 2^{k+1}} \left| \int_\Sigma \hat{g}(rx')\, d\sigma(x') \right| \leqslant C. \tag{2}$$

LEMMA 2. *There is a positive constant C such that for any* $(1, \infty)$-*atom f we can find $g \in H^1(R^n)$ such that $\hat{g} \geqslant |\hat{f}|$ on R^n and $g = \sum_{j=1}^\infty \alpha_j g_j$, where each g_j is a* $(1, 2)$-*atom centered at the origin and $\sum_{j=1}^\infty |\alpha_j| \leqslant C$.*

PROOF OF LEMMA 1. Let g be a $(1, 2)$-atom centered at the origin. Then for some $a > 0$

(i) g is supported in $B(0; a) = \{x : |x| \leqslant a\}$;

(ii) $\int_{B(0;a)} g(x)\, dx = 0$;

(iii) $\int_{B(0;a)} |g(x)|^2\, dx \leqslant C/a^n$.

For each integer k let $r_k \in [2^k, 2^{k+1}]$ be such that

$$\left| \int_\Sigma \hat{g}(r_k x')\, d\sigma(x') \right| = \sup_{2^k \leqslant r \leqslant 2^{k+1}} \left| \int_\Sigma \hat{g}(rx')\, d\sigma(x') \right|.$$

We will estimate the sum in (2) by establishing the inequalities

$$\sum_{2^k \cdot a < 1} \int_\Sigma |\hat{g}(r_k x')|\, d\sigma(x') \leqslant C \tag{3}$$

and

$$\sum_{2^k \cdot a \geqslant 1} \left| \int_\Sigma \hat{g}(r_k x')\, d\sigma(x') \right| \leqslant C. \tag{4}$$

For (3) we begin by noting that, because of (i) and (ii),

$$|\hat{g}(r_k x')| = \left| \int_0^a \int_\Sigma g(ry')(1 - e^{-2\pi i r r_k x' \cdot y'})\, d\sigma(y') r^{n-1}\, dr \right|.$$

Since (iii) implies $\int_{B(0;a)} |g(x)|\, dx \leqslant C$, and since

$$|1 - e^{-2\pi i r r_k x' \cdot y'}| \leqslant 2^{k+2} \cdot \pi \cdot a \qquad (0 \leqslant r \leqslant a),$$

it follows that $|\hat{g}(r_k x')| \leqslant C 2^k \cdot a$. This gives (3).

The estimate for (4) is a bit more complicated. Since

$$\int_\Sigma \hat{g}(r_k x')\, d\sigma(x') = \int_0^a \int_\Sigma g(ry') \int_\Sigma e^{-2\pi i r r_k x' \cdot y'}\, d\sigma(x') d\sigma(y') r^{n-1}\, dr,$$

it follows that (4) is bounded by

$$\int_0^a \int_\Sigma |g(ry')| \sum_{2^k \cdot a > 1} \left| \int_\Sigma e^{-2\pi i r r_k x' \cdot y'}\, d\sigma(x') \right| d\sigma(y') r^{n-1}\, dr. \tag{5}$$

We start by examining the numbers $|\int_\Sigma e^{-2\pi i r r_k x' \cdot y'}\, d\sigma(x')|$.

There is a constant C such that

$$\int_\Sigma e^{-itx' \cdot y'} d\sigma(x') = C \int_0^\pi e^{-it \cos\theta} \sin^{n-2}\theta\, d\theta$$

for $t > 0$ and $y' \in \Sigma$. (If $y' \in \Sigma$ is fixed, then the area of the $(n-2)$-dimensional sphere defined by $x' \cdot y' = \cos\theta$ is proportional to $\sin^{n-2}\theta$.) Since

$$\int_0^\pi e^{-it\cos\theta} \sin^{n-2}\theta\, d\theta = \int_{-1}^1 e^{ist} (1 - s^2)^{(n-3)/2}\, ds,$$

the proof of Lemma 3.11 in [4, p. 158] shows that

$$\left| \int_\Sigma e^{-itx' \cdot y'}\, d\sigma(x') \right| \leqslant C t^{(1-n)/2}, \qquad t > 0, y' \in \Sigma.$$

Using this inequality and remembering that $2^k \leqslant r_k$, we see that (5) is dominated by

$$C \int_0^a \int_\Sigma |g(ry')| \sum_{2^k \cdot a > 1} (2^k r)^{(1-n)/2} d\sigma(y') r^{n-1}\, dr.$$

By the Cauchy-Schwarz inequality, this is not greater than

$$C \left(\int_{B(0;\, a)} |g(x)|^2\, dx \right)^{1/2} \left\{ \int_0^a \left[\sum_{2^k \cdot a > 1} (2^k r)^{(1-n)/2} \right]^2 r^{n-1}\, dr \right\}^{1/2}. \tag{6}$$

Now (4) follows from (iii) and (6). Thus the proof of the lemma is complete.

The proof of Lemma 2 is patterned after a proof in [2] of the fact that for each $f \in H^1(R)$ we can find $g \in H^1(R)$ with $\hat{g} \geqslant |\hat{f}|$. We give the details only for the sake of completeness.

PROOF OF LEMMA 2. It is enough to prove the lemma for atoms $f \neq 0$ centered at the origin and such that $\hat{f}$ is real-valued. For $x = (x_1, \ldots, x_n) \in R^n$, put $\gamma(x) = \sum_{i=1}^n x_i^{2n}$. Then the "measure distance" $m(x, 0)$ in R^n is comparable to $\gamma^{1/2}(x)$. Since atoms are 3-molecules for $H^1(R^n)$, it follows that

$$\int_{R^n} |f(x)|^2\, dx \int_{R^n} |f(x)\gamma(x)|^2\, dx \leqslant C \tag{7}$$

for f as above (and with C independent of f). We will show that if $g \in L^2(R^n)$ is defined by $\hat{g} = |\hat{f}|$, then

$$\int_{R^n} |g(x)\gamma(x)|^2\, dx \leqslant \int_{R^n} |f(x)\gamma(x)|^2\, dx. \tag{8}$$

It will then follow from (7), the Plancherel theorem, and Theorem C of **[2]** that g has the required representation as a sum of atoms. (The atoms produced in the proof of Theorem C of **[2]** have the same center as the molecule which they compose.)

Now $\{x \in R^n: \hat{f}(x) = 0\}$ has Lebesgue measure 0 (since, e.g., $\hat{f}$ is the restriction to R^n of an entire function of n complex variables). Thus (8) will be established if we show

$$\left|\int_{R^n} g(x)\gamma(x)\overline{\phi(x)}\, dx\right| \leqslant \left(\int_{R^n} |f(x)\gamma(x)|^2\, dx\right)^{1/2} \left(\int_{R^n} |\phi(x)|^2\, dx\right)^{1/2} \tag{9}$$

whenever $\hat{\phi}$ is a smooth function with compact support contained in $\{x \in R^n: \hat{f}(x) \neq 0\}$.

Let $\Gamma = (2\pi i)^{-2n}\sum_{j=1}^n (\partial/\partial x_j)^{2n}$, so that the Fourier transform of $\gamma(x)h(x)$ is $\Gamma \hat{h}(x)$ for nice enough $h \in L^1(R^n)$. Then

$$\left|\int_{R^n} g(x)\overline{\gamma(x)\phi(x)}\, dx\right| = \left|\int_{R^n} \hat{g}(x)\overline{\Gamma\hat{\phi}(x)}\, dx\right|. \tag{10}$$

Since $\hat{g}$ is smooth on an open set in which $\hat{\phi}$ is supported, applying Fubini's theorem and integrating by parts shows that this last term is

$$\left|\int_{R^n} \Gamma\hat{g}(x)\overline{\hat{\phi}(x)}\, dx\right| \leqslant \int_{R^n} |\Gamma\hat{f}(x)| \cdot |\hat{\phi}(x)|\, dx. \tag{11}$$

Here the inequality follows from the fact that on each component of $\{x \in R^n: \hat{f}(x) \neq 0\}$, $\hat{g}$ coincides with either $\hat{f}$ or its negative. Finally, we estimate

$$\begin{aligned}\int_{R^n} |\Gamma\hat{f}(x)| \cdot |\hat{\phi}(x)|\, dx &\leqslant \left(\int_{R^n} |\Gamma\hat{f}(x)|^2\, dx\right)^{1/2} \left(\int_{R^n} |\hat{\phi}(x)|^2\, dx\right)^{1/2} \\ &= \left(\int_{R^n} |f(x)\gamma(x)|^2\, dx\right)^{1/2} \left(\int_{R^n} |\phi(x)|^2\, dx\right)^{1/2}.\end{aligned}$$

With (10) and (11), this establishes (9) and completes the proof.

3. An example. In this section we sketch a proof of the fact that (a′) and (b′) are not equivalent for nonradial functions $m(x)$ on R^n. The argument also shows that our Theorem must fail for $n = 1$. (These facts, and their proof, are well known to many people. We include them for completeness since they do not seem to have appeared in print.)

If (a′) and (b′) were equivalent, then the inequality

$$\sum_{k=-\infty}^{\infty} \sup\left\{|\hat{f}(x)|: 2^k \leqslant |x| \leqslant 2^{k+1}\right\} \leqslant C\|f\|_{H^1}, \quad f \in H^1(R^n), \tag{12}$$

would follow by a closed graph argument and duality. But (12) would imply, by the duality of H^1 and BMO, that the partial sums of any series

$$\sum_{k=-\infty}^{\infty} e^{2\pi i\langle x, x_k\rangle}, \qquad 2^k \leqslant |x_k| \leqslant 2^{k+1}, \tag{13}$$

are uniformly bounded in $BMO(R^n)$. This is false, since the partial sums of (13) are not even bounded uniformly in $L^2([0, 1]^n)$.

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