PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 1, January 1979

A COUNTEREXAMPLE TO A "THEOREM" ON L_n SETS[1]

JOHN D. BAILDON

ABSTRACT. An example is given of a closed connected set in E^r whose points of local nonconvexity can be decomposed into two convex subsets, but which is not arcwise connected and hence is not an L_n set. This contradicts a result by Valentine to which Stavrakas and Jamison have given a second proof. It is also shown that if the set of points of local nonconvexity of a closed connected set S in E^r can be decomposed into n *compact* subsets which are convex relative to S, then S is an L_{2n+1} set.

Introduction. In [2], Valentine gave three extensions of Tietze's theorem on convex sets, giving conditions under which a set should be an L_{n+1} or L_{2n+1} set. Stavrakas and Jamison in [1] produced a new proof for two of these results. We will give a counterexample to show that two of the theorems need more restrictive hypotheses and will prove such a new theorem using the method of Stavrakas and Jamison.

A set S is said to be an L_n set if every pair of points in S can be joined by a polygonal arc in S containing at most n segments.

A point $x \in S$ is a point of local nonconvexity of S if there does not exist a neighborhood N of x such that $N \cap S$ is convex.

A point x sees a point y via S (or x is visible to y via S) if the closed segment xy is contained in S.

The counterexample. The two results in [2] to which this example refers are

1. *Suppose S is a closed connected set in E^r. Furthermore, suppose that the set Q of all points of local nonconvexity of S can be decomposed into n convex subsets. Then S is an L_{2n+1} set.*

2. *Let S be a closed connected set in E^r. Suppose the set Q of all points of local nonconvexity of S can be decomposed into n connected closed subsets $Q_1, Q_2, \ldots, Q_n$ such that if $x \in Q_i$, $y \in Q_i$, $Q_i \in \{Q_1, Q_2, \ldots, Q_n\}$, then $xy \subset S$. Then S is an L_{2n+1} set.*

Consider the following sets in the plane: Let S_1 be the nonnegative x-axis, S_2 be the points on the line $y = 1$ for which $x \geqslant 1$, and S_3 be the union of the line segments from $(-1/n, 0)$ to $(n, 1)$ for all integers n. The set $S' = S_1 \cup S_2 \cup S_3$ is shown in Figure 1.

Received by the editors February 27, 1978 and, in revised form, April 24, 1978.

AMS (MOS) subject classifications (1970). Primary 52-00; Secondary 54G20.

Key words and phrases. Convex, locally convex, local nonconvexity, L_n set, Tietze's theorem.

[1]Support for this work was provided by Pennsylvania State University through grant 260-01SN PDE OCC EDUC PROG V.

0002-9939/79/0000-0019/$01.75

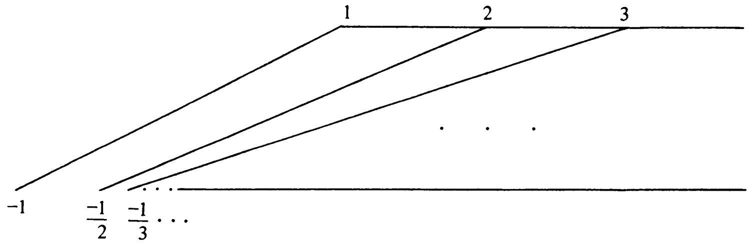

FIGURE 1

Let S_4 be the wedge in E^3 with S_2 as spine described by $\{(x, y, z)\colon z = |y - 1|;\ 0 \leqslant y \leqslant 2;\ x \geqslant 1\}$. The set $S = S' \cup S_4$ provides our counterexample. We note the following properties of S:

1. The set Q of points of local nonconvexity is the disjoint union of S_1 and S_2. These sets are convex and will serve as Q_1 and Q_2.
2. The set $S(Q_2)$ of points visible to Q_2 via S is not closed since each point of S_1 is a limit point of it which is not visible to Q_2.
3. S is connected, but not arcwise connected, since there is no arc from a point in Q_1 to one in Q_2.
4. In particular, there is no point of S seeing both Q_1 and Q_2.
5. Since S is not arcwise connected, it is not an L_n set for any n.

The modified theorem. It is instructive to see where the earlier proofs were in error. In [2], it was claimed that because S is closed and connected with every point seeing Q_1 or Q_2, it follows that there exists at least one point $y \in S$ and a pair of points $y_1 \in Q_1$, $y_2 \in Q_2$ such that y_i $(i = 1, 2)$ is a nearest point of Q_i which satisfies $yy_i \subset S$ $(i = 1, 2)$. The fourth remark after the counterexample indicates that this is not the case. In [1], the mistake in the proof lies in the assumption that the connected union of n L_3 sets will be an L_{3n} set. The two visibility sets $S(Q_1)$ and $S(Q_2)$ are each L_3 sets, but their union, S, is not an L_6 set, and in fact is not an L_n set for any n. It is a trivial matter to show that the connected union of n closed L_3 sets is an L_{3n} set. While our example shows that the visibility sets $S(Q_i)$ need not be closed, if we assume that the Q_i are compact, the closure of the $S(Q_i)$ is assured. This leads us to our modification of Valentine's result which is also an improvement in the sense that the Q_i are not required to be connected.

THEOREM. *Let S be a closed connected set in E^r. Suppose the set Q of all points of local nonconvexity of S can be decomposed into n compact subsets $Q_1, \ldots, Q_n$, such that if $x \in Q_i$, $y \in Q_i$, then $xy \subset S$. Then S is an L_{2n+1} set.*

We shall make use of the following lemmas from the papers in question:

LEMMA 1 (VALENTINE [2]). *Suppose S is a closed connected set which has at least one point of local nonconvexity. If x is a point of S, then there exists at least one point p of local nonconvexity of S such that $xp \subset S$.*

LEMMA 2 (STAVRAKAS AND JAMISON [1]). *Let S be a subset of E^r. Let x, $y \in S$ and suppose l is an arc from x to y of minimal arc length in S such that each point of l is a point of local convexity of S, except possibly for x and y. Then l is the closed line segment from x to y.*

PROOF OF THEOREM. By Lemma 1, every point of S is visible to some point of Q, so that $S = \bigcup_1^n S(Q_i)$ where $S(Q_i)$ denotes the set of points of S visible to a point in Q_i via S. Since the Q_i are compact, each $S(Q_i)$ is closed. Because of our hypothesis, each is also an L_3 set. Thus S is an L_{3n} set since S is connected; hence, given two points $x, y \in S$, there is an arc l between them of minimal length. Using the natural ordering on l, $l \cap Q_i$ has a first and last point (or is empty). Since l has minimal length, the hypothesis implies that $l \cap Q_i$ lies on a segment s_i. Thus $l - \bigcup_1^n s_i$ has at most $n + 1$ components and since $l - \bigcup_1^n s_i \subset l - Q$, each of the components is a segment by Lemma 2. Hence l consists of at most $2n + 1$ segments and S is an L_{2n+1} set.

COROLLARY. *Let S be a closed connected set in E^r. If the set Q of all points of local nonconvexity of S can be decomposed into n compact convex sets $Q_1, \ldots, Q_n$, then S is an L_{2n+1} set.*

EXAMPLE. The converse of the theorem is not true since the boundary of a square is an L_3 set, but the set of points of local nonconvexity (the vertices) do not form a single set Q of the type required, since the diagonally opposite vertices cannot see each other via S. The boundaries of appropriately large polygons will give counterexamples to the converse for larger n.

EXAMPLE. The theorem is an improvement over the corollary since we may get Q's as described which are not contained in the same number of convex subsets of S. Let S be the three vertical faces of a tetrahedron, so that Q will be the vertical edges. Then Q is not contained in a convex subset of S, but it has the property that $x, y \in Q$ implies $xy \subset S$.

EXAMPLE. A minimal arc in an L_n set is not necessarily an n-segment, as may be readily seen by considering a capital A. This is an L_2 set, but the minimal arc between two feet is a 3-segment, going across the bar of the letter.

QUESTION. Are there conditions under which minimal arcs in L_n sets will be n-segments?

The author wishes to thank the referee whose question as to why a union of L_3 sets would be an L_{3n} set caused the search for this counterexample.

BIBLIOGRAPHY

1. N. M. Stavrakas and R. E. Jamison, *Valentine's extensions of Tietze's theorem on convex sets*, Proc. Amer. Math. Soc. **36** (1972), 229–230.

2. F. A. Valentine, *Local convexity and L_n sets*, Proc. Amer. Math. Soc. **16** (1965), 1305–1310.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, WORTHINGTON SCRANTON CAMPUS, DUNMORE, PENNSYLVANIA 18512