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A COUNTEREXAMPLE IN l_2-MANIFOLD THEORY

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ABSTRACT. There is a space X such that $X \times X$ is l_2, but X is not l_2.

In [1], one asks whether $X \times X \approx l_2$ implies $X \approx l_2$. In this note, a counterexample to that question is given.

Let K be a noncontractible, compact holomogy cell. Then $Y = K \times l_2$ is a noncontractible l_2-manifold [4]. Define $X = \text{cone}(Y)$ with metric topology, and c its cone point $\{Y \times 0\}$. If α denotes the ordered pair (c, c) in $X \times X$, then $X - \{c\}$ and $X \times X - \{\alpha\}$ are l_2-manifolds. But, X is not since $\{c\}$ is not a Z-set in X. However,

THEOREM. $X \times X \approx l_2$.

PROOF. By Torunczyk's theorem [1, p. 127], it follows that the AR $X \times X$ is an l_2-manifold, since $\{\alpha\}$ is a Z-set in $X \times X$ by Lemma 2 below. Hence, $X \times X \approx l_2$ by Corollary 3 of [2]. $\square$

In the following, standard notion in infinite dimensional topology as Z-set, $l_2 \approx \prod_1^\infty]-1, 1[$, etc. . . . is referred to Chapman [1].

LEMMA 1. *$X \times X - \{\alpha\}$ is homeomorphic to l_2.*

PROOF. By Corollary 3 in [2], Lemma 1 will be proved by showing that $X \times X - \{\alpha\}$ is contractible. We intend to use Whitehead's theorem (Theorem 7.6.25 of [3]). So, two following properties need be verified.

(i) $\tilde{H}_*(X \times X - \{\alpha\}) = 0$.

Let $A = (X - \{c\}) \times X$ and $B = X \times (X - \{c\})$. Then, $A \cap B = (X - \{c\}) \times (X - \{c\})$ and $A \cup B = X \times X - \{\alpha\}$. Since $\tilde{H}_*(X - \{c\}) = \tilde{H}_*(K)$ is trivial, the Künneth formula [3, p. 227] shows that $\tilde{H}_*(A)$, $\tilde{H}_*(B)$ and $\tilde{H}_*(A \cap B)$ are trivial. Finally the Mayer-Vietoris sequence [3, p. 186], for open sets A and B of $X \times X - \{\alpha\}$, shows that $\tilde{H}_*(X \times X - \{\alpha\}) = 0$ as we desired.

(ii) *$X \times X - \{\alpha\}$ is simply connected.*

Given a map $f\colon S^1 \to X \times X - \{\alpha\}$, it suffices to show that f is null-homotopic in $X \times X - \{\alpha\}$. Let $f(t) = (h(t), g(t))$, where $h, g\colon S^1 \to X$ such that the compact subsets $h^{-1}(c)$ and $g^{-1}(c)$ of S^1 are disjoint.

Let $\{C_j | j = 1, \ldots, m\}$ be a finite family of pairwise disjoint arcs of S^1 missing $g^{-1}(c)$ such that the union of their interiors covers $h^{-1}(c)$.

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A null-homotopy of f in $X \times X - \{\alpha\}$ is the successive composition of three homotopies:

a. homotopes h (rel $S^1 - \bigcup_{j=1}^{m} \operatorname{Int} C_j$) in X to a map $h_{1/2}$ such that $h_{1/2}^{-1}(c) = \varnothing$.

b. homotopes g in X to a constant map g_1 in $X - \{c\}$.

c. homotopes $h_{1/2}$ in X to a constant map g_1 in $X - \{c\}$.

Now, $f = (h, g) \sim (h_{1/2}, g) \sim (h_{1/2}, g_1) \sim (h_1, g_1)$, where $\sim$ means homotopic to. The proof of Lemma 1 is compete.

LEMMA 2. *$\{\alpha\}$ is a Z-set in $X \times X$.*

PROOF. Given an open cover $\Omega = \{w_i\}$ of $X \times X$, it suffices to show that there is a map

$$f: X \times X \to X \times X - \{\alpha\}$$

which is Ω-close to $\mathrm{id}_{X \times X}$.

Let $Y_s = Y \times s \subset X$ where $0 < s < 1$ and $V_s = \operatorname{cone}(Y_s) = \{(y, t) \in X | t \leqslant s\}$. Then, there is an s $(0 < s < 1)$ such that the neighborhood $V_s \times V_s$ of α in $X \times X$ is contained in $w_{i_0} \in \Omega$.

It is obvious that $Z = (V_s \times Y_s) \cup (Y_s \times V_s)$ is the topological frontier of $V_s \times V_s$ in $X \times X$ and $V_s \times V_s - \{\alpha\} \approx X \times X - \{\alpha\}$ is an AR. Hence, the inclusion map $Z \to V_s \times V_s - \{\alpha\}$ extends to a map φ: $V_s \times V_s \to V_s \times V_s - \{\alpha\}$.

Now, we extend φ by the identity map over $X \times X - V_s \times V_s$ to obtain a desired Ω-approximation f of $\mathrm{id}_{X \times X}$. $\square$

REMARK. Replacing l_2 by $Q = \prod_i^{\infty}[-1, 1]$ in the above construction, we also obtain a compact X such that $X \times X \approx Q$ but $X \not\approx Q$.

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