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ORBITS OF PATHS UNDER HYPERBOLIC TORAL AUTOMORPHISMS

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ABSTRACT. A hyperbolic toral automorphism is a map f: $T^n \hookleftarrow$ such that has a linear lifting L: $\mathbf{R}^n \hookleftarrow$ without eigenvalues of modulus 1. In this note we prove that the orbit under f of a rectifiable nonconstant path γ: $[a, b] \to T^n$ contains a coset of a toral subgroup invariant under same power of f. For C^2 paths the same result was proved by J. Franks. For C^0 arcs S.G. Hancock proved that it is false.

A hyperbolic toral automorphism of $T^n = \mathbf{R}^n/\mathbf{Z}^n$ is a map that lifts to a linear map L: $\mathbf{R}^n \hookleftarrow$ without eigenvalues of modulus 1. Several authors ([1]–[6]) studied the invariant sets of these transformations. In [2] Franks proved that the closure of the orbit of a nonconstant C^2 path under a hyperbolic toral automorphism f contains a coset of a toral subgroup invariant under some power of f. S. G. Hancock in [3] showed that this property does not extend to C^0 paths. In this note we shall present a simple proof of an extension of Franks' result to rectifiable paths, i.e. continuous maps α: $[a, b] \to T^n$ such that there exists $K > 0$ satisfying $\Sigma_{n=0} d(\alpha(t_{n+1}), \alpha(t_n)) \leqslant K$ for all partition $a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{n+1} = b$, where $d(\cdot, \cdot)$ denotes any Riemannian metric on T^n.

THEOREM. *Let α: $(a, b) \to T^n$ be a rectifiable nonconstant path and f: $T^n \hookleftarrow$ a hyperbolic toral automorphism. Then the closure of the orbit of $\alpha((a, b))$ under f contains a coset of a toral subgroup invariant under some power of f.*

REMARK I. The toral subgroup cannot be required to be invariant under f. For instance if f_0: $T^2 \hookleftarrow$ is a hyperbolic toral automorphism define f: $T^2 \times T^2 \hookleftarrow$ as $f(x, y) = (y, f_0(x))$. This map is a hyperbolic automorphism without proper invariant toral subgroup. However the orbit of any path contained in $T^2 \times \{e\}$ is not dense in $T^2 \times T^2$.

REMARK II. If the hyperbolic toral automorphism f: $T^n \hookleftarrow$ is irreducible in the sense that there are no proper toral subgroups invariant under any power of f then our theorem clearly implies that the orbit of any nonconstant rectifiable path α: $[a, b] \to T^n$ is dense in T^n. It also implies the following stronger statement: $\alpha([a, b])$ contains transitive points (i.e. points with dense orbit). To see this let $J \subset [a, b]$ be the union of all intervals $(t_1, t_2) \subset [a, b]$ such that $\alpha(t) = \alpha(t_1)$ for all $t \in (t_1, t_2)$. Let $J' = [a, b] \setminus J$. J' is compact

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and nonempty. Let $\{U_n | n \in \mathbf{Z}\}$ be a basis of open neighborhoods and let $S_n = \{t \in J' | f^m(\alpha(t)) \in U_n$ for some $m\}$. Then S_n is open and it is dense because if $a \leqslant t_1 < t_2 \leqslant b$ belong to J' then $\alpha/[t_1, t_2]$ is nonconstant, hence its orbit is dense and $f^m(\alpha([t_1, t_2])) \cap U_n \neq 0$ for some m. Then every point in $\bigcap \{S_n | n \in \mathbf{Z}\}$ is transitive.

REMARK III. It is easy to construct a nonconstant C^0 path α: $[a, b] \to T^n$ whose orbit under an irreducible hyperbolic toral automorphism f is dense but that does not contain transitive points. Take a C^∞ path α: $[0, 1] \to T^n$ such that $\alpha(1/n) = e$ and $\alpha/[1/(n+1), 1/n]$ is nonconstant for all $n \geqslant 1$. Then the orbit of $\alpha/[1/(n+1), 1/n]$ is dense in T^n. By Hancock's results take α_n: $[1/(n+1), 1/n] \to T^n$ C^0 near to $\alpha/[1/(n+1), 1/n]$ such that $\alpha_n(1/n) = e$, $\alpha_n(1/(n+1)) = e$ and the orbit of $\alpha_n([1/(n+1), 1/n])$ is not dense. If α_n is near enough to $\alpha/[1/(n+1), 1/n]$ its orbit fills T^n up to $1/n$. Hence the orbit of the path $\bar{\alpha}$: $[0, 1] \to T^n$ defined by $\bar{\alpha}(t) = \alpha_n(t)$ if $t \in [1/(n+1), 1/n]$ and $\bar{\alpha}(0) = e$ is dense but the path does not contain transitive points.

PROOF OF THE THEOREM. Let π: $\mathbf{R}^n \to \mathbf{R}^n/\mathbf{Z}^n$ be the canonical covering map and $d(\cdot, \cdot)$ the flat distance on T^n. As usual define the length $l(\beta)$ of a path $\beta = [a, b] \to T^n$ (continuous or not) as the supremum of the sums $\sum_{n=0}^m d(\beta(t_{n+1}), \beta(t_n))$ over all partitions $a = t_0 \leqslant \cdots \leqslant t_{m+1} = b$. Define σ: $[a, b] \to \mathbf{R}$ by $\sigma(t) = l(\alpha/[a, t])$. This map is continuous (because α is continuous) and $\sigma(t_1) \geqslant \sigma(t_2)$ if $t_1 \geqslant t_2$. Let $[a', b'] = \sigma([a, b])$ and take τ: $[a', b'] \to [a, b]$ satisfying $\sigma\tau(t) = t$ for all $t \in [a, b]$. There always exists such a map (observe that we do not require τ to be continuous). Let α_0: $[a', b'] \to T^n$ defined by $\alpha_0 = \alpha\tau$. Then $l(\alpha_0/[t_1, t_2]) = t_2 - t_1$ for all $a' \leqslant t_1 < t_2 \leqslant b'$. In particular if $\varepsilon > 0$ and $[a_k, b_k]$, $k = 1, \ldots, m$, is a family of disjoint intervals contained in $[a', b']$ such that $\sum_0^m (b_k - a_k) \leqslant \varepsilon$ we have

$$\sum_0^m d(\alpha(a_k), \alpha(b_k)) \leqslant \sum_0^m l(\alpha_0/[a_k, b_k]) = \sum_0^m (b_k - a_k) \leqslant \varepsilon.$$

This proves that α_0 is an absolutely continuous path. Take γ: $[a', b'] \to \mathbf{R}^n$ absolutely continuous satisfying $\pi\gamma = \alpha_0$. Then, by well-known properties of absolutely continuous functions, there exists a set $E \subset [a, b]$ with total measure such that $\dot{\gamma}(t)$ exists for all $t \in E$ and:

$$\int_{t_1}^{t_2} \dot{\gamma}(s)\, ds = \gamma(t_2) - \gamma(t_1)$$

for all $a' \leqslant t_1 \leqslant t_2 \leqslant b'$. Now observe that we can suppose that the eigenvalues of the linear lifting L: $\mathbf{R}^n \hookleftarrow$ of f are either real and positive or are not roots of real numbers because if L does not satisfy this property we can replace f by a suitable power. Observe also that to prove the theorem it is sufficient to find an f-invariant toral subgroup M such that for all $\varepsilon > 0$ there exists a coset $y + M$ such that all its points can be ε-approximated by points in the orbit of $\gamma([a', b'])$. In order to find M we start defining the subspaces

E_λ as $\bigcup_{n>0}(L-\lambda I)^{-n}(0)$ if λ is real or

$$E_\lambda = \left(\left(\bigcup_{n>0}(\hat{L}-\lambda I)^{-n}(0)\right)\cup\left(\bigcup_{n>0}(\hat{L}-\bar{\lambda} I)^{-n}(0)\right)\right)\cap \mathbf{R}^n$$

where $\hat{L}$: $\mathbf{C}^n \hookleftarrow$, is the complexification of L, if λ is complex. Take eigenvalues $\lambda_1,\dots,\lambda_m$ of L such that $\dot{\gamma}(t)\in\bigoplus_{i=1}^m E_{\lambda_i}$ for all $t\in E$ and for all λ_j there exists $t\in E$ such that $\dot{\gamma}(t)$ has nonzero component on E_{λ_j}. The set $\lambda_1,\dots,\lambda_n$ is nonempty because γ is nonconstant. Suppose $|\lambda_1|\geqslant|\lambda_j|$ for all $1\leqslant j\leqslant m$. Take $t_0\in E$ such that $v=\dot{\gamma}(t_0)$ has nonzero projection on E_{λ_1}. Define $p(t)=\gamma(t)-\gamma(t_0)-(t-t_0)v$. Then $\lim_{t\to t_0}p(t)/(t-t_0)=0$ and

$$p(t)=\int_{t_0}^{t}\dot{\gamma}(s)\,ds-(t-t_0)v.$$

Hence:

$$p(t)\in\bigoplus_{i=1}^{m}E_{\lambda_i} \tag{0}$$

for all $t\in[a,b]$. Moreover since every eigenvalue of L has modulus $\neq 1$ we can assume $|\lambda_1|>1$ (otherwise replace f by f^{-1}). Define $\tilde{L}$: $S^{n-1}\hookleftarrow$ by $\tilde{L}w=Lw/\|Lw\|$. Using the real canonical form of L and the fact that every eigenvalue of L is either real and positive or has no real powers it is easy to prove that the limit Λ set of $v/\|v\|$ under $\tilde{L}$ is a connected, minimal set for $\tilde{L}$ and generates an L-invariant subspace S such that $\|Lw\|=|\lambda_1|\,\|w\|$ for all $w\in S$. Let M be the closure of $\pi(S)$. M is a toral subgroup. Take $w\in\Lambda$ such that $\pi(\{\lambda w|\ \lambda>0\})$ is dense in M. Given $\varepsilon>0$ take $T>0$ such that $d_0(\pi(\{\lambda w|\ 0<\lambda<T\}),\ M)<\varepsilon/3$, where $d_0(\cdot,\cdot)$ denotes the Hausdorff metric. Let $\delta>0$ be such that if $\bar{w}\in\mathbf{R}^n$, $\|\bar{w}\|=1$ and $\|\bar{w}-w\|\leqslant\delta$ then $d(\pi(\{\lambda\bar{w}|\ 0<\lambda<T\}),M)<\varepsilon/2$. Since the limit set of $v/\|v\|$ under $\tilde{L}$ is a minimal set for $\tilde{L}$ we can find an increasing sequence $\{n_k|\ k\geqslant 0\}$ of positive integers such that:

$$\lim_{k\to+\infty} n_k=\infty, \tag{1}$$

$$\|L^{n_k}v/\|L^{n_k}v\|-w\|<\delta,\qquad k\geqslant 0, \tag{2}$$

$$\sup\{n_{k+1}-n_k|\ k\geqslant 0\}<\infty. \tag{3}$$

From (3) we can define:

$$K=\sup\{\|L^{n_{k+1}}v\|/\|L^{n_k}v\|\ |\ k\geqslant 0\}. \tag{4}$$

Moreover there exist constants $C_1>0$, $C>0$ such that:

$$C_1\|L^n v\|\geqslant|\lambda_1|^n\ \|v\|,\qquad \|L^n w\|\leqslant C|\lambda_1|^n\ \|w\| \tag{5}$$

for all $w\in\bigoplus_{i=1}^m E_{\lambda_i}$, $n\geqslant 0$. Now choose $\delta_0>0$ satisfying

$$\|p(t)\|\leqslant\frac{\varepsilon}{2CC_1KT}\,|t-t_0|\ \|v\| \tag{6}$$

for all $|t - t_0| \leqslant \delta_0$. By (1) and (5) there exists n_k such that:

$$\|L^{n_k}(\delta_0 v)\| > T, \qquad \|L^{n_{k-1}}(\delta_0 v)\| \leqslant T. \tag{7}$$

The definition of K implies:

$$\|L^{n_k}(\delta_0 v)\| \leqslant KT. \tag{8}$$

We claim that:

$$\|L^{n_k}(\gamma(t) - \gamma(t_0)) - L^{n_k}(t - t_0)v\| \leqslant \varepsilon/2 \tag{9}$$

if $|t - t_0| \leqslant \delta_0$. By (0), (4), (5), (6) and (8):

$$\begin{aligned}
\|L^{n_k}(\gamma(t) - \gamma(t_0)) - L^{n_k}(t - t_0)v\| &= \|L^{n_k}p(t)\| \\
&\leqslant C|\lambda_1|^{n_k}\|p(t)\| \leqslant C|\lambda_1|^{n_k} \frac{\varepsilon}{2CC_1KT}|t - t_0|\ \|v\| \\
&\leqslant \frac{C\varepsilon}{2CC_1KT}|\lambda_1|^{n_k}\delta_0\|v\| \leqslant \frac{C\varepsilon}{2CC_1KT}C_1\|L^{n_k}(\delta_0 v)\| \\
&\leqslant \frac{C\varepsilon}{2CC_1KT}C_1KT = \frac{\varepsilon}{2}.
\end{aligned}$$

Now consider the coset $f^{n_k}(\alpha_0(t_0)) + M$. If $x \in M$ by (2) we can take $0 < t < T$ such that

$$d\left(\pi\left(t\frac{L^{n_k}v}{\|L^{n_k}v\|}\right), x\right) \leqslant \varepsilon/2.$$

Write $tL^{n_k}v/\|L^{n_k}v\| = L^{n_k}t_1 v$ with $t_1 = t/\|L^{n_k}v\|$. By (7) $0 < t_1 < \delta_0$. Then:

$$\begin{aligned}
&d\big(f^{n_k}(\alpha_0(t_0 + t_1)), f^{n_k}(\alpha(t_0)) + x\big) \\
&\qquad \leqslant \frac{\varepsilon}{2} + d\big(f^{n_k}(\alpha(t_0 + t_1)), f^{n_k}(\alpha(t_0)) + L^{n_k}t_1 v\big) \\
&\qquad \leqslant \frac{\varepsilon}{2} + \|L^{n_k}(\gamma(t_0 + t_1) - \gamma(t_0)) - L^{n_k}t_1 v\|
\end{aligned}$$

and using (9):

$$d\big(f^{n_k}(\alpha_0(t_0 + t_1)), f^{n_k}(\alpha(t_0)) + x\big) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore we found a coset of M such that all its points can be ε-approximated by points in the orbit of $\alpha_0([a', b'])$ as we wanted to prove.

REMARK IV. The theorem is also true for nonconstant paths α: $[a, b] \to T^n$ such that $\dot{\alpha}(t)$ exists for all $t \in [a, b]$. Even more general, the theorem is valid for nonconstant paths α: $[a, b] \to T^n$ with the following property: $(\mathcal{P}) - \alpha$ has a lifting γ: $[a, b] \to \mathbf{R}^n$ such that for every subspace G if there exist $a \leqslant t_1 \leqslant t_2 \leqslant b$ with $\gamma(t_1) - \gamma(t_2) \notin G$ then for some $t \in [a, b]$, $\dot{\gamma}(t)$ exists and does not belong to G.

In particular, paths α: $[a, b] \to T^n$ with derivative at every point satisfy this property because if γ: $[a, b] \to \mathbf{R}^n$ is a lifting of α, G a subspace and $\gamma(t_1) - \gamma(t_2) \notin G$ then taking linear maps φ_i: $\mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, k$, such that $\bigcap_{i=1}^{k}\varphi_i^{-1}(0) = G$ either $\dot{\gamma}(t) \notin G$ for some t or $\varphi_i\gamma$ is constant for all $i = 1, \ldots, k$. Hence $\varphi_i(\gamma(t_1) - \gamma(t_2)) = \varphi_i\gamma(t_1) - \varphi_i\gamma(t_2) = 0$ for all $i = 1, \ldots, k$ thus implying $\gamma(t_1) - \gamma(t_2) \in G$.

For paths satisfying ($\mathcal{P}$) the proof of the theorem is the same as previously with minor changes. Define $E \subset [a, b]$ as $E = \{t \in [a, b] \| \dot{\gamma}(t) \text{ exists}\}$. Then define $\bigoplus_{i=1}^{m} E_{\lambda_i}$, choose t_0 and define $p(t)$ as we did before (replacing γ: $[a', b'] \to \mathbf{R}^n$ by γ: $[a, b] \to \mathbf{R}^n$). To prove the relation $p(t) \in \bigoplus_{i=1}^{m} E_{\lambda_i}$ for all $t \in [a, b]$ observe that if $p(t) \notin \bigoplus_{i=1}^{m} E_{\lambda_i}$ for some $t \in [a, b]$ then $\gamma(t) - \gamma(t_0) = (t - t_0)v + p(t) \notin \bigoplus_{i=1}^{m} E_{\lambda_i}$. Hence for some $t_1 \in [a, b]$, $\dot{\gamma}(t_1)$ exists and does not belong to the subspace $\bigoplus_{i=1}^{m} E_{\lambda_i}$ thus contradicting its definition. After this modification the proof remains the same.

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