RECURSIVE DETERMINATION OF
THE SUM-OF-DIVISORS FUNCTION

JOHN A. EWELL

Abstract. A recursive scheme for determination of the sum-of-divisors function is presented. As all of the formulas involve triangular numbers, the scheme is therefore compared for efficiency with another known recursive triangular-number formula for this function.

For each positive integer $n$, $\sigma(n)$ denotes the sum of the positive divisors of $n$. Recently Ewell [1] derived two recursive schemes for computing the values $\sigma(n)$, and regarding efficiency of computation compared each of them with the well-known pentagonal-number formula of Euler. In the present discussion we derive another recursive scheme and show that among the four known recursive determinations of $\sigma$ this one ranks no worse than second with regard to efficiency. We prepare the way for an easy statement of our result by defining an auxiliary function $\alpha$ as follows. For each positive integer $n$, we express $n$ uniquely as $n = 2^{b(n)}O(n)$ where $b(n)$ is a nonnegative integer and $O(n)$ is odd. Now define $\alpha(n) = (2^{b(n)}+1 - 3)\sigma(O(n))$.

Theorem. For each positive integer $m$,

$$\sigma(2m - 1) - \sum_{k=0}^{\lfloor \frac{2m - 1}{2} \rfloor} \left\{ \alpha(2m - 1 - (2k + 1)(4k + 1)) + \alpha(2m - 1 - (2k + 1)(4k + 3)) \right\}$$

$$+ \sum_{k=0}^{m-1} \left\{ \sigma(2m - 1 - (2k + 2)(4k + 3)) + \sigma(2m - 1 - (2k + 2)(4k + 5)) \right\}$$

$$= \begin{cases} n(n + 1)/2, & \text{if } 2m - 1 = n(n + 1)/2, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

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and

\[\begin{align*}
\alpha(2m) - \sum_{k=0}^{\infty} \{ \sigma(2m - (2k + 1)(4k + 1)) + \sigma(2m - (2k + 1)(4k + 3)) \} \\
+ \sum_{k=0}^{\infty} \{ \alpha(2m - (2k + 2)(4k + 3)) + \alpha(2m - (2k + 2)(4k + 5)) \} \\
= \begin{cases} 
-n(n+1)/2, & \text{if } 2m = n(n+1)/2, \\
0, & \text{otherwise,}
\end{cases}
\end{align*}\]

(2)

where in both (1) and (2) summation extends as far as the arguments are positive.

**Proof.** Our proof depends on the following two identities due respectively to Euler and Gauss.

\[\begin{align*}
\prod_{n=1}^{\infty} (1 + x^n)(1 - x^{2n-1}) &= 1, \\
\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^n) &= \sum_{n=0}^{\infty} x^{n(n+1)/2}.
\end{align*}\]

For proofs see [2, pp. 277–284]. Since \(1 - x^{2n} = (1 - x^n)(1 + x^n)\), we use Euler’s identity to express Gauss’s identity in the equivalent form

\[\begin{align*}
\prod_{n=1}^{\infty} (1 - x^n) \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-2} &= \sum_{n=0}^{\infty} x^{n(n+1)/2}.
\end{align*}\]

Briefly, set \(F(x) = \sum x^{n(n+1)/2}\). Now take the logarithmic derivative of the foregoing identity and multiply the resulting identity by \(x\) to obtain

\[\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} - 2 \sum_{n=1}^{\infty} \frac{(2n - 1)x^{2n-1}}{1 - x^{2n-1}} = -x F'(x)/F(x) . \]

(3)

It is well known that the first series on the left of (3), a “Lambert” series, generates \(\sigma(n)\): i.e.,

\[\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n . \]

The second series is perhaps less well known, but straightforward algebraic manipulation shows that

\[\sum_{n=1}^{\infty} \frac{(2n - 1)x^{2n-1}}{1 - x^{2n-1}} = \sum_{n=1}^{\infty} x^n \sum_{d|n, d \text{ odd}} d \]

\[= \sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} + \sum_{m=1}^{\infty} \sigma(O(2m))x^{2m} . \]

Thus, identity (3) becomes

\[\sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} - \sum_{m=1}^{\infty} \sigma(2m)x^{2m} = x F'(x)/F(x) . \]
Now, separating the even and odd triangular numbers \( n(n + 1)/2 \) by the least positive residues of \( n \pmod 4 \), we write

\[
\left\{ \sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} \right\} F(x) = \sum_{m=1}^{\infty} x^{2m-1} \left\{ \sigma(2m - 1) + \sum_{k=0}^{\infty} \left[ \sigma(2m - 1 - (2k + 2)(4k + 3)) \right. \right. \\
+ \left. \sigma(2m - 1 - (2k + 2)(4k + 5)) \right] \right\}
\]

\[
+ \sum_{m=1}^{\infty} x^{2m} \left\{ \sum_{k=0}^{\infty} \left[ \sigma(2m - (2k + 1)(4k + 1)) \right. \right. \\
+ \left. \sigma(2m - (2k + 1)(4k + 3)) \right] \right\},
\]

and,

\[
\left\{ \sum_{m=1}^{\infty} \alpha(2m)x^{2m} \right\} F(x) = \sum_{m=1}^{\infty} x^{2m-1} \left\{ \sum_{k=0}^{\infty} \left[ \alpha(2m - 1 - (2k + 1)(4k + 1)) \right. \right. \\
+ \left. \alpha(2m - 1 - (2k + 1)(4k + 3)) \right] \right\}
\]

\[
+ \sum_{m=1}^{\infty} x^{2m} \left( \alpha(2m) + \sum_{k=0}^{\infty} \left[ \alpha(2m - (2k + 2)(4k + 3)) \right. \right. \\
+ \left. \alpha(2m - (2k + 2)(4k + 5)) \right] \right\}.
\]

We then substitute these last two developments into the identity

\[
\left\{ \sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} \right\} F(x) - \left\{ \sum_{m=1}^{\infty} \alpha(2m)x^{2m} \right\} F(x) = xF'(x) = \sum_{n=1}^{\infty} \frac{n(n + 1)}{2} x^{n(n+1)/2},
\]

equate coefficients of odd powers \( x^{2m-1} \) to obtain recurrence (1) and equate coefficients of even powers \( x^{2m} \) to obtain recurrence (2).

REMARKS. One of the three recurrences discussed in [1] is

\[
\sum_{k=0}^{\infty} (-1)^k (2k + 1)\sigma(n - k(k + 1)/2)
\]

\[
= \begin{cases} 
(-1)^{j+1} j(j + 1)(2j + 1)/6, & \text{if } n = j(j + 1)/2, \\
0, & \text{otherwise.}
\end{cases}
\]
Like our recurrences (1) and (2) this recurrence involves triangular numbers. Since \( \sigma \) is multiplicative and therefore \( \sigma(n) = (2^{b(n)}+1-1)\sigma(O(n)) \), we suppose that we are given a large odd number \( n \) and investigate efficiency of computation of \( \sigma(n) \) by recurrences (1) and (4). Theoretically we deduce that each of the recurrences needs about \( \sqrt{2n} \) of the values, \( \sigma(j) \), 1 \( < j < n \). But, practically, let us take a not-too-large value of \( n \), say \( n = 63 \), partially compute \( \sigma(63) \) by each of the recurrences and possibly observe some noteworthy differences.

By recurrence (1),

\[
\sigma(63) = (2^2 - 3)\sigma(31) + (2^5 - 3)\sigma(3) + (2^2 - 3)\sigma(9) + (2^3 - 3)\sigma(15) \\
+ (2^2 - 3)\sigma(21) + (2^4 - 3)\sigma(1) - \sigma(57) - \sigma(35) - \sigma(53) - \sigma(27).
\]

By recurrence (4),

\[
\sigma(63) = 3\sigma(62) - 5\sigma(60) + 7\sigma(57) - 9\sigma(53) + 11\sigma(48) - 13\sigma(42) \\
+ 15\sigma(35) - 17\sigma(27) + 19\sigma(18) - 21\sigma(8).
\]

Although each recurrence uses 10 lower values, we observe that recurrence (1) uses smaller values \( \sigma(j) \) by separating the binary and odd parts of \( j \). (This is something that any high-speed computing machine can do easily.) Also, recurrence (1) avoids coefficients such as the \( 2k + 1 \) of (4).

REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115