ON A THEOREM OF DUREN, SHAPIRO AND SHIELDS

SHINJI YAMASHITA

Abstract. We shall show that, an extension of the theorem of Duren, Shapiro and Shields on the univalence of a function $f$ holomorphic in the unit disk $D$, still remains of significance, if the power $\alpha \in (0, 1)$ in

$$\sup_{z \in D} (1 - |z|^2)^{\alpha} |f''(z)/f'(z)|$$

is small enough.

Let $f$ be a function nonconstant and holomorphic in $D = \{|z| < 1\}$, let $\alpha > 0$, and let

$$\|f\|_\alpha = \sup_{z \in D} (1 - |z|^2)^{\alpha} |f''(z)/f'(z)|$$

It was proved by P. L. Duren, H. S. Shapiro and A. L. Shields [3, Theorem 2] that if

$$\|f\|_1 < 2(\sqrt{5} - 2) = 0.47 \ldots,$$

then $f$ is univalent in $D$. This was improved later by J. Becker [1, Corollary 4.1, p. 36] in the sense that if $\|f\|_1 < 1$, then $f$ is univalent in $D$. Therefore, if $\alpha \in [0, 1]$, and if $\|f\|_\alpha < 1$, then $f$ is univalent in $D$ because $\|f\|_\alpha > \|f\|_1$.

We shall show, nevertheless, the method found by Duren, Shapiro and Shields still remains available for small $\alpha > 0$.

**Theorem.** Let $f$ be a function nonconstant and holomorphic in $D$. Suppose that $\alpha \in [0, 1]$, and suppose that

$$\|f\|_\alpha < 2(\sqrt{4^\alpha + 2 - \alpha - 2^\alpha}) \equiv k_\alpha.$$ 

Then $f$ is univalent in $D$.

Our theorem is thus significant for $k_\alpha > 1$; this is the case for $0 < \alpha < \alpha_0$, where $\alpha_0$ is the constant satisfying $2^{\alpha_0} + \alpha_0 = 7/4$. We note that $\alpha_0 = 0.416 \ldots$; the detailed calculation is due to the referee. The constant $k_1$ is the constant found by Duren, Shapiro and Shields.

We remark that the constant $k_0 = 2(\sqrt{3} - 1) = 1.46 \ldots$ is not good. In effect, it is known that if $\|f\|_0 < 2\sqrt{2} = 2.82 \ldots$, then $f$ is univalent in $D$; see [4, Problem 2, p. 179] (this Problem 2 is solved on making use of the Schwarz and Pick lemma, together with the theorem of V. V. Pokornyi, cited in [4, Problem 1, p. 179]).

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We begin with

**Lemma.** Let $g$ be a function holomorphic in $D$, and let $\alpha > 0$. Then

$$\sup_{z \in D} (1 - |z|^2)^{1+\alpha} |g'(z)| < 2^{1+\alpha} \sup_{z \in D} (1 - |z|^2)^{\alpha} |g(z)|.$$  \hspace{1cm} (1)

**Proof.** We may assume that the supremum in the right-hand side of (1), denoted by $M$, is finite, because the case where $M = +\infty$ is obvious. For each $z \in D$, we let

$$\rho = \left[ \frac{1}{2} (1 + r^2) \right]^{1/2}, \quad r = |z|.$$  

Then

$$(1 - r^2)^{1+\alpha} = 2^{1+\alpha} (\rho^2 - r^2) (1 - \rho^2)^{\alpha}.$$  

Since

$$g'(z) = (2\pi i)^{-1} \int_{|w| = \rho} g(w) (w - z)^{-2} \, dw,$$  

it follows that

$$(1 - |z|^2)^{1+\alpha} |g'(z)| < 2^{1+\alpha} (\rho^2 - r^2) (1 - \rho^2)^{\alpha} \left( \sup_{|w| = \rho} |g(w)| \right)$$

$$\times \left[ (2\pi)^{-1} \int_0^{2\pi} |pe^{i\theta} - z|^{-2} \rho \, d\theta \right]$$

$$< 2^{1+\alpha} M \left[ (2\pi)^{-1} \int_0^{2\pi} (\rho^2 - r^2) |pe^{i\theta} - z|^{-2} \, d\theta \right] = 2^{1+\alpha} M.$$  

We thus obtain (1).

**Proof of Theorem.** It follows from the Lemma, applied to $g = f''/f'$, that

$$(1 - |z|^2)^{1+\alpha} |(f''(z)/f'(z))'| < 2^{1+\alpha} \|f\|_{\alpha} < 2^{1+\alpha} k_{\alpha}, \quad z \in D.$$  

On the other hand, it follows from $1 + \alpha > 2\alpha$, that

$$(1 - |z|^2)^{1+\alpha} |f''(z)/f'(z)|^2 < \|f\|^2_{\alpha} < k_{\alpha}^2, \quad z \in D.$$  

Therefore, we obtain

$$ (1 - |z|^2)^{1+\alpha} |S_f(z)| < 2^{1+\alpha} k_{\alpha} + \frac{1}{2} k_{\alpha}^2 = 2(2 - \alpha), \quad z \in D, \hspace{1cm} (2)$$

where $S_f = (f''/f')' - \frac{1}{2} (f''/f')^2$ is the Schwarzian derivative of $f$.

Now, P. R. Beesack [2, (2.6), p. 217, and (2.8), p. 218] proved that if there exists $\lambda \in [1, 2]$ such that

$$(1 - |z|^2)^{\lambda} |S_f(z)| < 2(3 - \lambda), \quad z \in D,$$  

then $f$ is univalent in $D$. Since $1 < 1 + \alpha < 2$, it follows from (2) that $f$ is univalent in $D$.

**References**


Department of Mathematics, Tokyo Metropolitan University, Fukazawa, Setagaya-ku, Tokyo 158, Japan