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ON A THEOREM OF DUREN, SHAPIRO AND SHIELDS

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ABSTRACT. We shall show that, an extension of the theorem of Duren, Shapiro and Shields on the univalence of a function f holomorphic in the unit disk D, still remains of significance, if the power $\alpha \in (0, 1)$ in

$$\sup_{z \in D}(1 - |z|^2)^\alpha |f''(z)/f'(z)|$$

is small enough.

Let f be a function nonconstant and holomorphic in $D = \{|z| < 1\}$, let $\alpha \geqslant 0$, and let

$$\|f\|_\alpha = \sup_{z \in D}(1 - |z|^2)^\alpha |f''(z)/f'(z)|.$$

It was proved by P. L. Duren, H. S. Shapiro and A. L. Shields [**3**, Theorem 2] that if

$$\|f\|_1 \leqslant 2(\sqrt{5} - 2) = 0.47\ldots,$$

then f is univalent in D. This was improved later by J. Becker [**1**, Corollary 4.1, p. 36] in the sense that if $\|f\|_1 \leqslant 1$, then f is univalent in D. Therefore, if $\alpha \in [0, 1]$, and if $\|f\|_\alpha \leqslant 1$, then f is univalent in D because $\|f\|_\alpha \geqslant \|f\|_1$.

We shall show, nevertheless, the method found by Duren, Shapiro and Shields still remains available for small $\alpha > 0$.

THEOREM. *Let f be a function nonconstant and holomorphic in D. Suppose that $\alpha \in [0, 1]$, and suppose that*

$$\|f\|_\alpha \leqslant 2(\sqrt{4^\alpha + 2 - \alpha} - 2^\alpha) \equiv k_\alpha.$$

Then f is univalent in D.

Our theorem is thus significant for $k_\alpha > 1$; this is the case for $0 \leqslant \alpha < \alpha_0$, where α_0 is the constant satisfying $2^{\alpha_0} + \alpha_0 = 7/4$. We note that $\alpha_0 = 0.416\ldots$; the detailed calculation is due to the referee. The constant k_1 is the constant found by Duren, Shapiro and Shields.

We remark that the constant $k_0 = 2(\sqrt{3} - 1) = 1.46\ldots$ is not good. In effect, it is known that if $\|f\|_0 \leqslant 2\sqrt{2} = 2.82\ldots$, then f is univalent in D; see [**4**, Problem 2, p. 179] (this Problem 2 is solved on making use of the Schwarz and Pick lemma, together with the theorem of V. V. Pokornyi, cited in [**4**, Problem 1, p. 179]).

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We begin with

LEMMA. *Let g be a function holomorphic in D, and let $\alpha \geqslant 0$. Then*

$$\sup_{z\in D}(1-|z|^2)^{1+\alpha}|g'(z)| \leqslant 2^{1+\alpha}\sup_{z\in D}(1-|z|^2)^{\alpha}|g(z)|. \tag{1}$$

PROOF. We may assume that the supremum in the right-hand side of (1), denoted by M, is finite, because the case where $M = +\infty$ is obvious. For each $z \in D$, we let

$$\rho = \left[\tfrac{1}{2}(1+r^2)\right]^{1/2}, \qquad r = |z|.$$

Then

$$(1-r^2)^{1+\alpha} = 2^{1+\alpha}(\rho^2 - r^2)(1-\rho^2)^{\alpha}.$$

Since

$$g'(z) = (2\pi i)^{-1}\int_{|w|=\rho} g(w)(w-z)^{-2}\,dw,$$

it follows that

$$\begin{aligned}(1-|z|^2)^{1+\alpha}|g'(z)| &\leqslant 2^{1+\alpha}(\rho^2-r^2)(1-\rho^2)^{\alpha}(\sup_{|w|=\rho}|g(w)|)\\ &\quad\times\left[(2\pi)^{-1}\int_0^{2\pi}|\rho e^{i\theta}-z|^{-2}\rho\,d\theta\right]\\ &\leqslant 2^{1+\alpha}M\left[(2\pi)^{-1}\int_0^{2\pi}(\rho^2-r^2)|\rho e^{i\theta}-z|^{-2}\,d\theta\right] = 2^{1+\alpha}M.\end{aligned}$$

We thus obtain (1).

PROOF OF THEOREM. It follows from the Lemma, applied to $g = f''/f'$, that

$$(1-|z|^2)^{1+\alpha}|(f''(z)/f'(z))'| \leqslant 2^{1+\alpha}\|f\|_\alpha \leqslant 2^{1+\alpha}k_\alpha, \quad z \in D.$$

On the other hand, it follows from $1+\alpha \geqslant 2\alpha$, that

$$(1-|z|^2)^{1+\alpha}|f''(z)/f'(z)|^2 \leqslant \|f\|_\alpha^2 \leqslant k_\alpha^2, \quad z \in D.$$

Therefore, we obtain

$$(1-|z|^2)^{1+\alpha}|S_f(z)| \leqslant 2^{1+\alpha}k_\alpha + \tfrac{1}{2}k_\alpha^2 = 2(2-\alpha), \quad z \in D, \tag{2}$$

where $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$ is the Schwarzian derivative of f.

Now, P. R. Beesack [2, (2.6), p. 217, and (2.8), p. 218] proved that if there exists $\lambda \in [1, 2]$ such that

$$(1-|z|^2)^{\lambda}|S_f(z)| \leqslant 2(3-\lambda), \quad z \in D,$$

then f is univalent in D. Since $1 \leqslant 1+\alpha \leqslant 2$, it follows from (2) that f is univalent in D.

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