COMPACTIFICATION OF A CONVERGENCE SPACE

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ABSTRACT. A characterization for the class of convergence spaces having the largest Hausdorff compactification is given, and regularity and \( \lambda \)-Hausdorffness of modified Richardson compactification are discussed.

Introduction. A convergence space, though a generalisation of a topological space, may behave quite differently from a topological space, e.g. unlike a Hausdorff topological space every Hausdorff convergence space has a Hausdorff compactification which, of course, need not be the largest one [6]. The question when a Hausdorff convergence space has the largest Hausdorff compactification seems to have considerable importance. In [4] and [5], necessary and sufficient conditions for a Hausdorff convergence space \((X, q_X)\) to have the largest Hausdorff compactification are found; it is observed in [8] that the proof of the necessity part is not sound and shown that the largest Hausdorff compactification of \((X, q_X)\), whenever it exists, is given by the modified Richardson compactification \((X^*, q_{X^*})\); using this we find the largest class of Hausdorff convergence spaces having the largest Hausdorff compactification. We also discuss when \((X^*, q_{X^*})\) is regular for a regular Hausdorff convergence space \((X, q_X)\), and when for a topological Tychonoff space \((X, q_X), (X^*, q_{X^*})\) and \((\lambda X^*, q_{\lambda X^*})\), the topological modification of \((X^*, q_{X^*})\) [7], are homeomorphic to \(\beta X\), the topological Stone-Cech compactification of \((X, q_X)\).

Definitions and notations. For definitions, not given here, the reader is asked to refer to [2] and [7]. We shall follow the notations of [8]. For a set \(X\), \(FX\) denotes the set of all filters on \(X\) and \(PX\) the set of all subsets of \(X\). Let u.f. denote ultrafilter. For \(x \in X\), \(x = \{ A \subset X| x \in A \}\) is the principal filter containing \(x\). A convergence structure (c.s.) on \(X\) is a function \(q_X\) from \(FX\) to \(PX\) satisfying the following conditions:

1. for \(x \in X\), \(x \in q_X(x)\);
2. for \(\varphi, \psi \in FX\), if \(\varphi \subset \psi\), then \(q_X(\varphi) \subset q_X(\psi)\);
3. if \(x \in q_X(\varphi)\), then \(x \in q_X(\varphi \cap x)\).

The pair \((X, q_X)\) is called a convergence space. If \(x \in q_X(\varphi)\), we say that \(\varphi\) is
convergent (cgt) and converges to x and \( x = \lim \varphi \). If \( q_x(\varphi) \) is empty, \( \varphi \) is called nonconvergent (noncgt). For \( A \subseteq X \), \( A \) is called open if every \( \varphi \in FX \) and converging to \( x \in A \) contains \( A \); \( A \) is called closed if \( X - A \) is open; \( A \) is called almost compact if every u.f. containing \( A \) is cgt; closure of \( A \) to be denoted by

\[
\text{cl}(A, q_x) = \{ x \in X | x \in q_x(\varphi) \text{ for } \varphi \in FX \text{ and } A \subseteq \varphi \}.
\]

For \( \varphi \in FX \), \( \text{cl}(\varphi, q_x) \) (in short \( \text{cl} \varphi \) when there is no loss of clarity) is the filter generated by \( \{ \text{cl}(A, q_x) | A \subseteq \varphi \} \). Let \( \varphi, \psi \in FX \), then \( \varphi \) and \( \psi \) are called 0-distinct if there exist \( A \subseteq \varphi \) and \( B \subseteq \psi \) such that \( A \) and \( B \) are open and \( A \cap B = \emptyset \).

A convergence space is called (a) \( \lambda \)-Hausdorff if \((\lambda X, q_X)\) is Hausdorff (b) essentially compact if it has only finitely many noncgt u.f.'s (c) almost locally compact if every cgt filter contains an almost compact set. Let \( f : X \to Y \) be a function. For \( \varphi \in FX \), \( f_{\varphi} \) is the filter \( \{ B \subseteq Y | f^{-1}B \subseteq \varphi \} \). Henceforward the word “space” is used to mean a Hausdorff convergence space. We shall not distinguish between equivalent sets and also between homeomorphic spaces. A compactification of a space \((X, q_X)\) will mean a Hausdorff compactification and will be denoted by a triple \((X^1, q_{X^1}, 1_X)\), where \((X^1, q_{X^1})\) is a compact space and \(1_X: (X, q_X) \to (X^1, q_{X^1})\) an embedding. An embedding will be treated as an inclusion function. If \((X^1, q_{X^1}, 1_X)\) and \((X^2, q_{X^2}, 2_X)\) are two compactifications of \((X, q_X)\), then we say that \((X^1, q_{X^1}, 1_X) \supseteq (X^2, q_{X^2}, 2_X)\) if there exists a continuous map \( f: (X^1, q_{X^1}) \to (X^2, q_{X^2})\) such that \( f \circ 1_X = 2_X \). Call a compactification \((X^1, q_{X^1}, 1_X)\) of \((X, q_X)\) universal if given a compact space \((Y, q_Y)\) and a continuous map \( f: (X, q_X) \to (Y, q_Y) \), there exists a continuous map \( g: (X^1, q_{X^1}) \to (Y, q_Y) \) such that \( g \circ 1_X = f \). If \((X, q_X)\) is a topological Tychonoff space, then \( \beta X \) will denote its topological Stone-Čech compactification.

\( H\text{-Conv} \) denotes the category of spaces and continuous maps. \( CH\text{-Conv}, EH\text{-Conv} \) and \( ALH\text{-Conv} \) denote the subcategories of \( H\text{-Conv} \) consisting of compact spaces, essentially compact spaces and almost locally compact spaces respectively. A full subcategory \( \mathcal{B} \) of a full subcategory \( \mathcal{C} \) of \( H\text{-Conv} \) is called embedding epireflective in \( \mathcal{C} \) if \( \mathcal{B} \) is epireflective in \( \mathcal{C} \) and each reflection map is a dense embedding.

For a space \((X, q_X)\) and \( A \subseteq X \), define \( \hat{A} \) and \( A^* \) as \( \hat{A} = \{ \varphi \in FX | A \subseteq \varphi \) and \( \varphi \) is a noncgt u.f.\) and \( A^* = A \cup \hat{A} \). For \( \varphi \in FX \), \( \varphi' \) is the filter \( \{ K \subseteq X^* | K \cap X \in \varphi \} \) on \( X^* \) and \( \varphi^* \) the filter generated by \( \{ A^* | A \subseteq \varphi \} \) on \( X^* \). If \( \varphi \in X^* \), then \( \varphi^0 \) is the filter generated by \( \{ A \cup (\varphi) | A \subseteq \varphi \} \) on \( X^* \). For \( \phi \in FX^* \), \( \phi_0 \) is the filter \( \{ A | A^* \subseteq \phi \} \) on \( X \) and \( \phi_0 \) is the set \( \{ K \subseteq X | K \subseteq \phi \} \).

In [6], for a space \((X, q_X)\), a c.s. \( q_{X^*}^* \) is defined on \( X^* \) as follows. Let \( \phi \in FX^* \). For \( x \in X, x \in q_{X^*}^*(\phi) \) iff \( x \in q_X(\phi^*) \), and for \( \varphi \in X^* \), \( \varphi \in q_{X^*}^*(\phi) \) iff \( \varphi^* \subseteq \phi \). Then \( (X^*, q_{X^*}^*, \cdot_X^*), \) where \( \cdot_X^* \) is inclusion from \( X \) to \( X^* \), becomes the Richardson compactification of \((X, q_X)\). Another c.s. \( q_{X^*} \) is defined on \( X^* \).
in [8] as follows. Let \( \phi \in FX^* \). For \( x \in X \), \( x \in q_{X^*}(\phi) \) iff \( x \in q_X(\phi_*) \), and for \( \varphi \in \hat{X} \), \( \varphi \in q_{X^*}(\phi) \) iff there exists an u.f. \( \Psi \) on \( X^* \) such that \( \varphi^* \subset \Psi \) and \( \phi \cap \Psi \subset \phi \). Following [8] we shall refer to \((X^*, q_{X^*}, \cdot^*)\), where \( \cdot^* \) is inclusion from \( X \) to \( X^* \), as the modified Richardson compactification of \( (X, q_X) \).

For \( \psi \in \hat{X} \), define a c.s. \( q_{X^*}^\psi \) on \( X^* \) as follows. Let \( \phi \in FX^* \). For \( x \in X \), \( x \in q_{X^*}^\phi(\phi) \) iff \( x \in q_X(\phi_*) \); for \( \varphi \in \hat{X} \), if \( \varphi \neq \psi \), then \( \varphi \in q_{X^*}^\phi(\phi) \) iff \( \varphi' \cap \hat{\psi} \subset \phi \), and \( \psi \in q_{X^*}^\phi(\phi) \) iff either \( \varphi' \cap \hat{\psi} \subset \phi \) or there exists an u.f. \( \Psi \) on \( X^* \) such that \( \hat{X} \in \Psi \) and \( \Psi \cap \hat{\psi} \subset \phi \). Let \( \cdot^* \) denote the inclusion from \( X \) to \( X^* \). If \( (X, q_X) \) is almost locally compact, then \((X^*, q_{X^*}^\cdot, \cdot^*)\) becomes a compactification of \( (X, q_X) \); call \((X^*, q_{X^*}^\cdot, \cdot^*)\) a point-fixed-modified-Richardson compactification of \( (X, q_X) \).

Call a compactification \((X^1, q_{X^1}, 1_X)\) of a space \((X, q_X)\) Richardson type (in short, \( R \)-type) if \( |X^1| = |X^*| \) and for \( \phi \in FX^1 \), \( x \in X \), \( x \in q_{X^1}(\phi) \) iff \( x \in q_X(\phi_*) \), and for \( \varphi \in \hat{X} \), \( \varphi \in q_{X^1}(1_X \varphi) \).

1.

1.1 Lemma [5]. If \((X^1, q_{X^1}, 1_X)\) is a largest compactification of a space \((X, q_X)\), then \( X \) is open in \((X^1, q_{X^1})\).

Proof. Let \((X^2, q_{X^2}, s_X)\) be one point compactification of \((X, q_X)\) given in [3]. Then there exists a continuous map

\[
g: (X^1, q_{X^1}) \to (X^2, q_{X^2})
\]

such that \( gx = x \) for \( x \in X \). This implies that \( X = X^1 - g^{-1}(X^2 - X) \) is open in \((X^1, q_{X^1})\).

Let \((Y, q_Y)\) be a regular compact space and \( f: (X, q_X) \to (Y, q_Y) \) a continuous map. Define a function

\[
\bar{f}: (X^*, q_{X^*}) \to (Y, q_Y)
\]

as follows. For \( x \in X \), \( \bar{f}x = fx \) and for \( \varphi \in \hat{X} \), \( \bar{f}\varphi = \lim f\varphi \). It can be seen that for \( \phi \in FX^* \), \( \cl f\phi_* \subset f\phi \). Hence \( \bar{f} \) is continuous. Thus we have proved

1.2 Proposition. Every continuous map \( f: (X, q_X) \to (Y, q_Y) \), where \((Y, q_Y)\) is a regular compact space, has a continuous extension \( \bar{f}: (X^*, q_{X^*}) \to (Y, q_Y) \).

1.3 Proposition. A space \((X, q_X)\) is almost locally compact iff \( X \) is open in \((X^*, q_{X^*})\).

Proof. If \((X, q_X) \in ALH-Conv\), then \( X \) is open in \((X^*, q_{X^*})\) because for every \( \phi \in FX^* \) converging to \( x \in X \), \( \phi = \varphi' \) for some \( \varphi \in FX \). Now suppose that \( \phi \in FX \) converges to \( x \in X \). Since every u.f. on \( X^* \) containing \( \varphi^* \) converges to \( x \), \( X \in q^* \). Hence there exists \( A \in \varphi \) such that \( A^* \subset X \) implying that \( A \) is almost compact.

1.4 Corollary. If a space is almost locally compact, then it is open in each of its \( R \)-type compactifications.
1.5 Lemma. Let \((X^1, q_{X^1}, 1_X)\) and \((X^2, q_{X^2}, 2_X)\) be two compactifications of \((X, q_X)\) such that \((X^1, q_{X^1}, 1_X) > (X^2, q_{X^2}, 2_X)\). If \((X^2, q_{X^2}, 2_X)\) is R-type, then \(|X^1| = |X^2|\) and \(\text{id}: (X^1, q_{X^1}) \to (X^2, q_{X^2})\) is continuous; if, in addition, \((X, q_X)\) is almost locally compact, then \((X^1, q_{X^1}, 1_X)\) is also R-type.

Proof. Since \((X^1, q_{X^1}, 1_X) > (X^2, q_{X^2}, 2_X)\), there exists a continuous map \(f: (X^1, q_{X^1}) \to (X^2, q_{X^2})\) such that \(f \circ 1_X = 2_X\). Let \(t \in X^1 - X\). There exists an u.f. \(\varphi\) on \(X\) such that \(1_X \varphi\) converges to \(t\). Clearly \(ft = \varphi\), and \(f\) is 1-1 and onto. Hence \(|X^1| = |X^2|\). Also, \(f\) can be treated as the identity function. Now suppose that \((X^1, q_{X^1}) \in \text{ALH-Conv}\). Let \(\phi \in FX^1\) and \(x \in X\). If \(x \in q_{X^1}(\phi)\), then \(x \in q_X(\phi_\ast)\). If \(x \in q_X(\phi_\ast)\), then \(x \in q_{X^1}(1_X \phi_\ast)\) and hence \(x \in q_{X^1}(\phi)\), because \(\phi_\ast = \phi_0\).

1.6 Theorem. If \((X, q_X)\) is almost locally compact, then for every \(\psi \in \hat{X}\), \((X^\ast, q_{X^\ast}, \ast)\) is a maximal compactification of \((X, q_X)\).

Proof. Let \((X^1, q_{X^1}, 1_X)\) be a compactification of \((X, q_X)\) such that \((X^1, q_{X^1}, 1_X) > (X^\ast, q_{X^\ast}, \ast)\). By 1.5, \((X^1, q_{X^1}, 1_X)\) is R-type. Now to prove that \((X^\ast, q_{X^\ast}, \ast) > (X^1, q_{X^1}, 1_X)\), it suffices to prove that for an u.f. \(\phi\) on \(X^\ast\) containing \(\hat{X}, \psi \in q_{X^1}(\phi)\), which is clear in view of 1.5.

1.7 Proposition. For an almost locally compact space \((X, q_X)\), \((X^\ast, q_{X^\ast}) = (X^\ast, q_X^\ast)\) for every \(\psi \in \hat{X}\) iff \((X, q_X)\) is essentially compact.

Proof. Let \((X^\ast, q_{X^\ast}) = (X^\ast, q_X^\ast)\) for every \(\psi \in \hat{X}\). If there exists a free u.f. \(\phi\) on \(X^\ast\) containing \(\hat{X}\), then \(\psi \in q_{X^\ast}(\phi)\) for every \(\psi \in \hat{X}\), which is not true unless \(X\) is singleton, in which case \(\phi\) is not free. Hence \((X, q_X) \in \text{EH-Conv}\). Conversely, if \((X, q_X) \in \text{EH-Conv}\), then no free u.f. on \(X^\ast\) containing \(\hat{X}\) exists and hence \((X^\ast, q_{X^\ast}) = (X^\ast, q_X^\ast)\) for every \(\psi \in \hat{X}\).

1.8 Proposition. If a space \((X, q_X)\) has a largest compactification, then it is essentially compact.

Proof. If \((X, q_X)\) has a largest compactification, then, by Theorem 12 of [8], it is given by the modified Richardson compactification \((X^\ast, q_{X^\ast}, \ast)\). By 1.1 and 1.3, \((X, q_X) \in \text{ALH-Conv}\). Now in view of 1.6, \((X^\ast, q_{X^\ast}) = (X^\ast, q_X^\ast)\) for every \(\psi \in \hat{X}\). Hence \((X, q_X) \in \text{EH-Conv}\) by 1.7.

1.9 Theorem. A space has a universal compactification iff it is essentially compact.

Proof. Apply 1.8 and Proposition 5 of [8].

The following is a categorical version of 1.9.

1.10. Theorem. \(\text{CH-Conv}\) is embedding epireflective in \(\text{EH-Conv}\), and if \(\mathcal{C}\) is the largest full subcategory of \(\text{H-Conv}\) such that \(\text{CH-Conv}\) is embedding epireflective in \(\mathcal{C}\), then \(\mathcal{C} = \text{EH-Conv}\).

Proof. Obvious.
2.

2.1 **Lemma.** For a space $(X, q_X)$ and $A \subset X$, 
\[
cl(A, q_{X^*}) = cl(A^*, q_{X^*}) = cl(A, q_X) \cup \hat{A}.
\]

**Proof.** Obvious.

2.2 **Lemma.** The following are equivalent for a space $(X, q_X)$:
(i) $\varphi^0 = \varphi^*$ and $\varphi = cl(\varphi, q_X)$ for all $\varphi \in \hat{X}$;
(ii) $\varphi^0 = cl(\varphi^0, q_{X^*})$ for all $\varphi \in \hat{X}$.

**Proof.** (i) implies (ii). Let $A \cup \{\varphi\} \in \varphi^0$ for $\varphi \in \hat{X}$. By (i), there exists $B \in \varphi$ such that $cl(B, q_X) \subset A$ and $B^* \subset A \cup \{\varphi\}$ implying that $cl(B \cup \{\varphi\}, q_{X^*}) \subset A \cup \{\varphi\}$ by 2.1. Hence $\varphi^0 = cl(\varphi^0, q_{X^*})$.

(ii) implies (i). Let $A \in \varphi$ for $\varphi \in \hat{X}$. Since $A \cup \{\varphi\} \in \varphi^0$, there exists $B \in \varphi$ such that 
\[
cl(B \cup \{\varphi\}, q_{X^*}) \subset A \cup \{\varphi\}.
\]
This implies that $B^* \subset A \cup \{\varphi\}$ and $cl(B, q_X) \subset A$. Hence $\varphi^0 = \varphi^*$ and $\varphi = cl(\varphi, q_X)$.

2.3 **Theorem.** For a regular space $(X, q_X)$, $(X^*, q_{X^*})$ is regular iff $\varphi^0 = cl(\varphi^0, q_{X^*})$ for every noncgt u.f. $\varphi$ on $X$.

**Proof.** Suppose that $(X^*, q_{X^*})$ is regular. Since for $\varphi \in \hat{X}$, $\varphi^0$ converges to $\varphi$, $cl(\varphi^0, q_{X^*})$ converges to $\varphi$. Now it can be seen that $\varphi^0 = cl(\varphi^0, q_{X^*})$. Conversely, to prove that $(X^*, q_{X^*})$ is regular, let $\phi$ be a cgt filter on $X^*$. If $\phi$ converges to $x \in X$, then in view of 2.1, $cl(\phi^* \subset (cl \phi)_*)$. Hence $cl \phi$ converges to $x$. If $\phi$ converges to $\varphi \in \hat{X}$, then because of the given condition and 2.2 either $\varphi^0 \subset \phi$ or $\phi = \hat{\phi}$; in both the cases $cl \phi$ converges to $\varphi$.

2.4 **Corollary.** If $(X^*, q_{X^*})$ is regular, then $(X^*, q_{X^*}) = (X^*, q_X^*)$.

2.5 **Theorem.** If $(X, q_X)$ is topological, then the following are equivalent:
(i) $(X^*, q_{X^*})$ is topological;
(ii) $(X, q_X)$ is Tychonoff and $(X^*, q_{X^*}) = \beta X$;
(iii) $\varphi^0 = cl(\varphi^0, q_{X^*})$ for every noncgt u.f. $\varphi$ on $X$.

**Proof.** (i) implies (ii). Apply 1.2.

(ii) implies (iii). Apply 2.3.

(iii) implies (i). Apply 2.3 and Corollary 2 of [1].

2.6 **Corollary.** If $(X^*, q_{X^*})$ is topological, then so is $(X^*, q_X^*)$.

If $WX$ and $\beta X$ respectively denote the Wallman compactification and Fomin $H$-closed extension of a topological space $(X, q_X)$, then combining 2.6 and Theorem 3 of [1] we get

2.7 **Theorem.** If $(X^*, q_{X^*})$ is topological, then $(X, q_X)$ is normal and $\beta X = WX = \beta X$. 

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2.8 THEOREM. If \((X^*, q_{X^*}) = \beta X, K \subset \beta X - X\) is closed in \(\beta X\) and \(X_1 = X \cup K\) is an extension of \(X\) in \(\beta X\), then \((X_1^*, q_{X_1}) = \beta X_1\).

PROOF. By 2.6 \((X^*, q_{X^*}) = \beta X\) and so \((X_1^*, q_{X_1}) = \beta X_1\) by Theorem 2 of [1]. \(K\) being closed in \(\beta X\), it can be easily verified that \(|\hat{X}| = |\hat{X}_1|\) and \(\alpha = \text{cl}(\alpha, q_{X^*})\) for all \(\alpha \in \hat{X}_1\). Hence \((X_1^*, q_{X_1}) = \beta X_1\).

2.9 PROPOSITION. For a space \((X, q_X)\), the following are true:

(i) \((\lambda X^*, q_{\lambda X^*}) = (\lambda X^*, q_{\lambda X^*})\);

(ii) \((A^*|A \subset X\text{ and } A \text{ open in } X)\) is a base for the topology of \((\lambda X^*, q_{\lambda X^*})\).

PROOF. (i) Since for every u.f. \(\phi\) on \(X^*\),

\[ q_{X^*}(\phi) = q_{X^*}(\phi), \quad (\lambda X^*, q_{\lambda X^*}) = (\lambda X^*, q_{\lambda X^*}). \]

(ii) For \(A \subset X\) and \(A\) open in \(X\), \(A^*\) is clearly open in \((\lambda X^*, q_{\lambda X^*})\). For \(K\) open in \((\lambda X^*, q_{\lambda X^*})\), it can be seen that \(K = U \{B^*|B^* \subset K\text{ and } B \text{ open in } X\} \).

2.10 THEOREM. The following are equivalent for a topological space \((X, q_X)\):

(i) \((X, q_X)\) is Tychonoff and \((\lambda X^*, q_{\lambda X^*}) = \beta X\);

(ii) Any two nonct u.f.'s on \((X, q_X)\) are 0-distinct;

(iii) \((X^*, q_{X^*})\) is \(\lambda\)-Hausdorff.

PROOF. (i) implies (ii). Let \(\varphi, \psi \in \hat{X}\). If \(\varphi\) and \(\psi\) are not 0-distinct, then \(\{A^* \cap B^*|A, B \text{ open in } X\text{ and } A \in \varphi, B \in \varphi\}\) will generate a filter on \(X^*\) converging to both \(\varphi\) and \(\psi\) in \((\lambda X^*, q_{\lambda X^*})\).

(ii) implies (iii) and (iii) implies (i) are obvious.

REMARK. Since even for a topological space \((X, q_X)\), \((X^*, q_{X^*})\) is 'highly nontopological' in the sense that not many filters other than u.f.'s are cgt in \((X^*, q_{X^*})\), it is more interesting to know when is \((\lambda X^*, q_{\lambda X^*}) = \beta X\) than to know when is \((X^*, q_{X^*}) = \beta X\).

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