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COMPACTIFICATION OF A CONVERGENCE SPACE

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ABSTRACT. A characterization for the class of convergence spaces having the largest Hausdorff compactification is given, and regularity and λ-Hausdorffness of modified Richardson compactification are discussed.

Introduction. A convergence space, though a generalisation of a topological space, may behave quite differently from a topological space, e.g. unlike a Hausdorff topological space every Hausdorff convergence space has a Hausdorff compactification which, of course, need not be the largest one [6]. The question when a Hausdorff convergence space has the largest Hausdorff compactification seems to have considerable importance. In [4] and [5], necessary and sufficient conditions for a Hausdorff convergence space (X, q_X) to have the largest Hausdorff compactification are found; it is observed in [8] that the proof of the necessity part is not sound and shown that the largest Hausdorff compactification of (X, q_X), whenever it exists, is given by the modified Richardson compactification (X^*, q_{X^*}); using this we find the largest class of Hausdorff convergence spaces having the largest Hausdorff compactification. We also discuss when (X^*, q_{X^*}) is regular for a regular Hausdorff convergence space (X, q_X), and when for a topological Tychonoff space (X, q_X), (X^*, q_{X^*}) and $(\lambda X^*, q_{\lambda X^*})$, the topological modification of (X^*, q_{X^*}) [7], are homeomorphic to βX, the topological Stone-Čech compactification of (X, q_X).

Definitions and notations. For definitions, not given here, the reader is asked to refer to [2] and [7]. We shall follow the notations of [8]. For a set X, FX denotes the set of all filters on X and PX the set of all subsets of X. Let u.f. denote ultrafilter. For $x \in X$, $\dot{x} = \{A \subset X | x \in A\}$ is the principal filter containing $\{x\}$. A *convergence structure* (c.s.) on X is a function q_X from FX to PX satisfying the following conditions:

(1) for $x \in X$, $x \in q_X(\dot{x})$;

(2) for $\varphi, \psi \in FX$, if $\varphi \subset \psi$, then $q_X(\varphi) \subset q_X(\psi)$;

(3) if $x \in q_X(\varphi)$, then $x \in q_X(\varphi \cap \dot{x})$.

The pair (X, q_X) is called a *convergence space*. If $x \in q_X(\varphi)$, we say that φ is

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convergent (cgt) and converges to x and $x = \lim \varphi$. If $q_X(\varphi)$ is empty, φ is called nonconvergent (noncgt). For $A \subset X$, A is called *open* if every $\varphi \in FX$ and converging to $x \in A$ contains A; A is called *closed* if $X - A$ is open; A is called *almost compact* if every u.f. containing A is cgt; *closure* of A to be denoted by

$$\mathrm{cl}(A, q_X) = \{x \in X | x \in q_X(\varphi) \text{ for } \varphi \in FX \text{ and } A \in \varphi\}.$$

For $\varphi \in FX$, $\mathrm{cl}(\varphi, q_X)$ (in short $\mathrm{cl}\,\varphi$ when there is no loss of clarity) is the filter generated by $\{\mathrm{cl}(A, q_X) | A \in \varphi\}$. Let $\varphi, \psi \in FX$, then φ and ψ are called 0-distinct if there exist $A \in \varphi$ and $B \in \psi$ such that A and B are open and $A \cap B = \varnothing$.

A convergence space is called (a) λ-Hausdorff if $(\lambda X, q_{\lambda X})$ is Hausdorff (b) *essentially compact* if it has only finitely many noncgt u.f.'s (c) *almost locally compact* if every cgt filter contains an almost compact set. Let f: $X \to Y$ be a function. For $\varphi \in FX$, $f\varphi$ is the filter $\{B \subset Y | f^{-1}B \in \varphi\}$. Henceforward the word "space" is used to mean a Hausdorff convergence space. We shall not distinguish between equivalent sets and also between homeomorphic spaces. A compactification of a space (X, q_X) will mean a Hausdorff compactification and will be denoted by a triple $(X^1, q_{X^1}, 1_X)$, where (X^1, q_{X^1}) is a compact space and 1_X: $(X, q_X) \to (X^1, q_{X^1})$ an embedding. An embedding will be treated as an inclusion function. If $(X^1, q_{X^1}, 1_X)$ and $(X^2, q_{X^2}, 2_X)$ are two compactifications of (X, q_X), then we say that $(X^1, q_{X^1}, 1_X) \geqslant (X^2, q_{X^2}, 2_X)$ if there exists a continuous map f: $(X^1, q_{X^1}) \to (X^2, q_{X^2})$ such that $f \circ 1_X = 2_X$. Call a compactification $(X^1, q_{X^1}, 1_X)$ of (X, q_X) universal if given a compact space (Y, q_Y) and a continuous map f: $(X, q_X) \to (Y, q_Y)$, there exists a continuous map g: $(X^1, q_{X^1}) \to (Y, q_Y)$ such that $g \circ 1_X = f$. If (X, q_X) is a topological Tychonoff space, then βX will denote its topological Stone-Čech compactification.

H-Conv denotes the category of spaces and continuous maps. CH-Conv, EH-Conv and ALH-Conv denote the subcategories of H-Conv consisting of compact spaces, essentially compact spaces and almost locally compact spaces respectively. A full subcategory $\mathfrak{B}$ of a full subcategory $\mathfrak{A}$ of H-Conv is called *embedding epireflective* in $\mathfrak{A}$ if $\mathfrak{B}$ is epireflective in $\mathfrak{A}$ and each reflection map is a dense embedding.

For a space (X, q_X) and $A \subset X$, define $\hat{A}$ and A^* as $\hat{A} = \{\varphi \in FX | A \in \varphi$ and φ is a noncgt u.f.$\}$ and $A^* = A \cup \hat{A}$. For $\varphi \in FX$, φ' is the filter $\{K \subset X^* | K \cap X \in \varphi\}$ on X^* and φ^* the filter generated by $\{A^* | A \in \varphi\}$ on X^*. If $\varphi \in \hat{X}$, then φ^0 is the filter generated by $\{A \cup \{\varphi\} | A \in \varphi\}$ on X^*. For $\phi \in FX^*$, ϕ_* is the filter $\{A | A^* \in \phi\}$ on X and ϕ_0 is the set $\{K \cap X | K \in \phi\}$.

In [6], for a space (X, q_X), a c.s. $q^*_{X^*}$ is defined on X^* as follows. Let $\phi \in FX^*$. For $x \in X$, $x \in q^*_{X^*}(\phi)$ iff $x \in q_X(\phi_*)$, and for $\varphi \in \hat{X}$, $\varphi \in q^*_{X^*}(\phi)$ iff $\varphi^* \subset \phi$. Then $(X^*, q^*_{X^*}, *_X)$, where $*_X$ is inclusion from X to X^*, becomes the Richardson compactification of (X, q_X). Another c.s. q_{X^*} is defined on X^*

in [8] as follows. Let $\phi \in FX^*$. For $x \in X$, $x \in q_{X^*}(\phi)$ iff $x \in q_X(\phi_*)$, and for $\varphi \in \hat{X}$, $\varphi \in q_{X^*}(\phi)$ iff there exists an u.f. Ψ on X^* such that $\varphi^* \subset \Psi$ and $\dot{\varphi} \cap \Psi \subset \phi$. Following [8] we shall refer to $(X^*, q_{X^*}, *_X)$, where $*_X$ is inclusion from X to X^*, as the modified Richardson compactification of (X, q_X).

For $\psi \in \hat{X}$, define a c.s. $q^{\psi}_{X^*}$ on X^* as follows. Let $\phi \in FX^*$. For $x \in X$, $x \in q^{\psi}_{X^*}(\phi)$ iff $x \in q_X(\phi_*)$; for $\varphi \in \hat{X}$, if $\varphi \neq \psi$, then $\varphi \in q^{\psi}_{X^*}(\phi)$ iff $\varphi' \cap \dot{\varphi} \subset \phi$, and $\psi \in q^{\psi}_{X^*}(\phi)$ iff either $\psi' \cap \dot{\psi} \subset \phi$ or there exists an u.f. Ψ on X^* such that $\hat{X} \in \Psi$ and $\Psi \cap \dot{\psi} \subset \phi$. Let $*_X$ denote the inclusion from X to X^*. If (X, q_X) is almost locally compact, then $(X^*, q^{\psi}_{X^*}, *_X)$ becomes a compactification of (X, q_X); call $(X^*, q^{\psi}_{X^*}, *_X)$ a point-fixed-modified-Richardson compactification of (X, q_X).

Call a compactification $(X^1, q_{X^1}, 1_X)$ of a space (X, q_X) Richardson type (in short, R-type) if $|X^1| = |X^*|$ and for $\phi \in FX^1$ and $x \in X$, $x \in q_{X^1}(\phi)$ iff $x \in q_X(\phi_*)$, and for $\varphi \in \hat{X}$, $\varphi \in q_{X^1}(1_X\varphi)$.

1.

1.1 Lemma [5]. *If $(X^1, q_{X^1}, 1_X)$ is a largest compactification of a space (X, q_X), then X is open in (X^1, q_{X^1}).*

Proof. Let (X^s, q_{X^s}, s_X) be one point compactification of (X, q_X) given in [3]. Then there exists a continuous map

$$g\colon (X^1, q_{X^1}) \to (X^s, q_{X^s})$$

such that $gx = x$ for $x \in X$. This implies that $X = X^1 - g^{-1}(X^s - X)$ is open in (X^1, q_{X^1}).

Let (Y, q_Y) be a regular compact space and $f\colon (X, q_X) \to (Y, q_Y)$ a continuous map. Define a function

$$\bar{f}\colon (X^*, q_{X^*}) \to (Y, q_Y)$$

as follows. For $x \in X$, $\bar{f}x = fx$ and for $\varphi \in \hat{X}$, $\bar{f}\varphi = \lim f\varphi$. It can be seen that for $\phi \in FX^*$, $\operatorname{cl} f\phi_* \subset \bar{f}\phi$. Hence $\bar{f}$ is continuous. Thus we have proved

1.2 Proposition. *Every continuous map $f\colon (X, q_X) \to (Y, q_Y)$, where (Y, q_Y) is a regular compact space, has a continuous extension $\bar{f}\colon (X^*, q_{X^*}) \to (Y, q_Y)$.*

1.3 Proposition. *A space (X, q_X) is almost locally compact iff X is open in (X^*, q_{X^*}).*

Proof. If $(X, q_X) \in ALH$-Conv, then X is open in (X^*, q_{X^*}) because for every $\phi \in FX^*$ converging to $x \in X$, $\phi = \varphi'$ for some $\varphi \in FX$. Now suppose that $\varphi \in FX$ converges to $x \in X$. Since every u.f. on X^* containing φ^* converges to x, $X \in \varphi^*$. Hence there exists $A \in \varphi$ such that $A^* \subset X$ implying that A is almost compact.

1.4 Corollary. *If a space is almost locally compact, then it is open in each of its R-type compactifications.*

1.5 LEMMA. *Let* $(X^1, q_{X^1}, 1_X)$ *and* $(X^2, q_{X^2}, 2_X)$ *be two compactifications of* (X, q_X) *such that* $(X^1, q_{X^1}, 1_X) \geqslant (X^2, q_{X^2}, 2_X)$. *If* $(X^2, q_{X^2}, 2_X)$ *is R-type, then* $|X^1| = |X^2|$ *and* id: $(X^1, q_{X^1}) \to (X^2, q_{X^2})$ *is continuous; if, in addition,* (X, q_X) *is almost locally compact, then* $(X^1, q_{X^1}, 1_X)$ *is also R-type.*

PROOF. Since $(X^1, q_{X^1}, 1_X) \geqslant (X^2, q_{X^2}, 2_X)$, there exists a continuous map f: $(X^1, q_{X^1}) \to (X^2, q_{X^2})$ such that $f \circ 1_X = 2_X$. Let $t \in X^1 - X$. There exists an u.f. φ on X such that $1_X\varphi$ converges to t. Clearly $ft = \varphi$, and f is 1-1 and onto. Hence $|X^1| = |X^2|$. Also, f can be treated as the identity function. Now suppose that $(X, q_X) \in ALH$-Conv. Let $\phi \in FX^1$ and $x \in X$. If $x \in q_{X^1}(\phi)$, then $x \in q_X(\phi_*)$. If $x \in q_X(\phi_*)$, then $x \in q_{X^1}(1_X\phi_*)$ and hence $x \in q_{X^1}(\phi)$, because $\phi_* = \phi_0$.

1.6 THEOREM. *If* (X, q_X) *is almost locally compact, then for every* $\psi \in \hat{X}$, $(X^*, q^{\psi}_{X^*}, *_X)$ *is a maximal compactification of* (X, q_X).

PROOF. Let $(X^1, q_{X^1}, 1_X)$ be a compactification of (X, q_X) such that $(X^1, q_{X^1}, 1_X) \geqslant (X^*, q^{\psi}_{X^*}, *_X)$. By 1.5, $(X^1, q_{X^1}, 1_X)$ is R-type. Now to prove that $(X^*, q^{\psi}_{X^*}, *_X) \geqslant (X^1, q_{X^1}, 1_X)$, it suffices to prove that for an u.f. ϕ on X^* containing $\hat{X}$, $\psi \in q_{X^1}(\phi)$, which is clear in view of 1.5.

1.7 PROPOSITION. *For an almost locally compact space* (X, q_X), $(X^*, q_{X^*}) = (X^*, q^{\psi}_{X^*})$ *for every* $\psi \in \hat{X}$ *iff* (X, q_X) *is essentially compact.*

PROOF. Let $(X^*, q_{X^*}) = (X^*, q^{\psi}_{X^*})$ for every $\psi \in \hat{X}$. If there exists a free u.f. ϕ on X^* containing $\hat{X}$, then $\psi \in q_{X^*}(\phi)$ for every $\psi \in \hat{X}$, which is not true unless X is singleton, in which case ϕ is not free. Hence $(X, q_X) \in EH$-Conv. Conversely, if $(X, q_X) \in EH$-Conv, then no free u.f. on X^* containing $\hat{X}$ exists and hence $(X^*, q_{X^*}) = (X^*, q^{\psi}_{X^*})$ for every $\psi \in \hat{X}$.

1.8 PROPOSITION. *If a space* (X, q_X) *has a largest compactification, then it is essentially compact.*

PROOF. If (X, q_X) has a largest compactification, then, by Theorem 12 of **[8]**, it is given by the modified Richardson compactification $(X^*, q_{X^*}, *_X)$. By 1.1 and 1.3, $(X, q_X) \in ALH$-Conv. Now in view of 1.6, $(X^*, q_{X^*}) = (X^*, q^{\psi}_{X^*})$ for every $\psi \in \hat{X}$. Hence $(X, q_X) \in EH$-Conv by 1.7.

1.9 THEOREM. *A space has a universal compactification iff it is essentially compact.*

PROOF. Apply 1.8 and Proposition 5 of **[8]**.

The following is a categorical version of 1.9.

1.10. THEOREM. CH-Conv *is embedding epireflective in* EH-Conv, *and if* $\mathcal{Q}$ *is the largest full subcategory of* H-Conv *such that* CH-Conv *is embedding epireflective in* $\mathcal{Q}$, *then* $\mathcal{Q} = EH$-Conv.

PROOF. Obvious.

2.

2.1 LEMMA. *For a space* (X, q_X) *and* $A \subset X$,

$$\mathrm{cl}(A, q_{X^*}) = \mathrm{cl}(A^*, q_{X^*}) = \mathrm{cl}(A, q_X) \cup \hat{A}.$$

PROOF. Obvious.

2.2 LEMMA. *The following are equivalent for a space* (X, q_X):
(i) $\varphi^0 = \varphi^*$ *and* $\varphi = \mathrm{cl}(\varphi, q_X)$ *for all* $\varphi \in \hat{X}$;
(ii) $\varphi^0 = \mathrm{cl}(\varphi^0, q_{X^*})$ *for all* $\varphi \in \hat{X}$.

PROOF. (i) implies (ii). Let $A \cup \{\varphi\} \in \varphi^0$ for $\varphi \in \hat{X}$. By (i), there exists $B \in \varphi$ such that $\mathrm{cl}(B, q_X) \subset A$ and $B^* \subset A \cup \{\varphi\}$ implying that $\mathrm{cl}(B \cup \{\varphi\}, q_{X^*}) \subset A \cup \{\varphi\}$ by 2.1. Hence $\varphi^0 = \mathrm{cl}(\varphi^0, q_{X^*})$.

(ii) implies (i). Let $A \in \varphi$ for $\varphi \in \hat{X}$. Since $A \cup \{\varphi\} \in \varphi^0$, there exists $B \in \varphi$ such that

$$\mathrm{cl}(B \cup \{\varphi\}, q_{X^*}) \subset A \cup \{\varphi\}.$$

This implies that $B^* \subset A \cup \{\varphi\}$ and $\mathrm{cl}(B, q_X) \subset A$. Hence $\varphi^0 = \varphi^*$ and $\varphi = \mathrm{cl}(\varphi, q_X)$.

2.3 THEOREM. *For a regular space* (X, q_X), (X^*, q_{X^*}) *is regular iff* $\varphi^0 = \mathrm{cl}(\varphi^0, q_{X^*})$ *for every noncgt u.f.* φ *on* X.

PROOF. Suppose that (X^*, q_{X^*}) is regular. Since for $\varphi \in \hat{X}$, φ^0 converges to φ, $\mathrm{cl}(\varphi^0, q_{X^*})$ converges to φ. Now it can be seen that $\varphi^0 = \mathrm{cl}(\varphi^0, q_{X^*})$. Conversely, to prove that (X^*, q_{X^*}) is regular, let ϕ be a cgt filter on X^*. If ϕ converges to $x \in X$, then in view of 2.1, $\mathrm{cl}\,\phi_* \subset (\mathrm{cl}\,\phi)_*$. Hence $\mathrm{cl}\,\phi$ converges to x. If ϕ converges to $\varphi \in \hat{X}$, then because of the given condition and 2.2 either $\varphi^0 \subset \phi$ or $\phi = \dot{\varphi}$; in both the cases $\mathrm{cl}\,\phi$ converges to φ.

2.4 COROLLARY. *If* (X^*, q_{X^*}) *is regular, then* $(X^*, q_{X^*}) = (X^*, q^*_{X^*})$.

2.5 THEOREM. *If* (X, q_X) *is topological, then the following are equivalent*:
(i) (X^*, q_{X^*}) *is topological*;
(ii) (X, q_X) *is Tychonoff and* $(X^*, q_{X^*}) = \beta X$;
(iii) $\varphi^0 = \mathrm{cl}(\varphi^0, q_{X^*})$ *for every noncgt u.f.* φ *on* X.

PROOF. (i) implies (ii). Apply 1.2.
(ii) implies (iii). Apply 2.3.
(iii) implies (i). Apply 2.3 and Corollary 2 of [1].

2.6 COROLLARY. *If* (X^*, q_{X^*}) *is topological, then so is* $(X^*, q^*_{X^*})$.

If WX and $\mathfrak{F}X$ respectively denote the Wallman compactification and Fomin H-closed extension of a topological space (X, q_X), then combining 2.6 and Theorem 3 of [1] we get

2.7 THEOREM. *If* (X^*, q_{X^*}) *is topological, then* (X, q_X) *is normal and* $\beta X = WX = \mathfrak{F}X$.

2.8 THEOREM. *If* $(X^*, q_{X^*}) = \beta X$, $K \subset \beta X - X$ *is closed in* βX *and* $X_1 = X \cup K$ *is an extension of* X *in* βX, *then* $(X_1^*, q_{X_1^*}) = \beta X_1$.

PROOF. By 2.6 $(X^*, q_{X^*}^*) = \beta X$ and so $(X_1^*, q_{X_1^*}^*) = \beta X_1$ by Theorem 2 of [1]. K being closed in βX, it can be easily verified that $|\hat{X}| = |\hat{X}_1|$ and $\alpha = \mathrm{cl}(\alpha, q_{X_1})$ for all $\alpha \in \hat{X}_1$. Hence $(X_1^*, q_{X_1^*}) = \beta X_1$.

2.9 PROPOSITION. *For a space* (X, q_X), *the following are true*:

(i) $(\lambda X^*, q_{\lambda X^*}) = (\lambda X^*, q_{\lambda X^*}^*)$;

(ii) $\{A^* | A \subset X$ *and* A *open in* $X\}$ *is a base for the topology of* $(\lambda X^*, q_{\lambda X^*})$.

PROOF. (i) Since for every u.f. ϕ on X^*,

$$q_{X^*}(\phi) = q_{X^*}^*(\phi), \quad (\lambda X^*, q_{\lambda X^*}) = (\lambda X^*, q_{\lambda X^*}^*).$$

(ii) For $A \subset X$ and A open in X, A^* is clearly open in $(\lambda X^*, q_{\lambda X^*})$. For K open in $(\lambda X^*, q_{\lambda X^*})$, it can be seen that $K = \bigcup \{B^*: B^* \subset K$ and B open in $X\}$.

2.10 THEOREM. *The following are equivalent for a topological space* (X, q_X):

(i) (X, q_X) *is Tychonoff and* $(\lambda X^*, q_{\lambda X^*}) = \beta X$;

(ii) *Any two noncgt u.f.'s on* (X, q_X) *are* 0-*distinct*;

(iii) (X^*, q_{X^*}) *is* λ-*Hausdorff*.

PROOF. (i) implies (ii). Let $\varphi, \psi \in \hat{X}$. If φ and ψ are not 0-distinct, then $\{A^* \cap B^* | A, B$ open in X and $A \in \varphi$, $B \in \varphi\}$ will generate a filter on X^* converging to both φ and ψ in $(\lambda X^*, q_{\lambda X^*})$.

(ii) implies (iii) and (iii) implies (i) are obvious.

REMARK. Since even for a topological space (X, q_X), (X^*, q_{X^*}) is 'highly nontopological' in the sense that not many filters other than u.f.'s are cgt in (X^*, q_{X^*}), it is more interesting to know when is $(\lambda X^*, q_{\lambda X^*}) = \beta X$ than to know when is $(X^*, q_{X^*}) = \beta X$.

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REFERENCES

1. R. J. Gazik, *Regularity of Richardson's compactification*, Canad. J. Math. **26** (1974), 1289–1293.

2. D. C. Kent and G. D. Richardson, *Minimal convergence spaces*, Trans. Amer. Math. Soc. **160** (1971), 487–499.

3. C. J. M. Rao, *On smallest compactification for convergence spaces*, Proc. Amer. Math. Soc. **44** (1974), 225–230.

4. _____, *On largest Hausdorff compactification for convergence spaces*, Bull. Austral. Math. Soc. **12** (1975), 73–79.

5. _____, *On extremal compactifications of convergence spaces*, Ph.D. Thesis, Indian Institute of Technology, Kanpur, 1975.

6. G. D. Richardson, *A Stone-Čech compactification for limit spaces*, Proc. Amer. Math. Soc. **25** (1970), 403–404.

7. G. D. Richardson and D. C. Kent, *Regular compactifications of convergence spaces*, Proc. Amer. Math. Soc. 31 (1972), 571–573.

8. Vinod-Kumar, *On the largest Hausdorff compactification of a Hausdorff convergence space*, Bull. Austral. Math. Soc. 16 (1977), 189–197.

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