SHORTER NOTES

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A SHORT PROOF OF A VERSION OF ASPLUND'S NORM AVERAGING THEOREM

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Abstract. A short proof is given of a somewhat weaker version of Asplund's result on averaging smooth and rotund norms in Banach spaces.

In 1967 E. Asplund [1] found a general construction, which, in the case of locally uniformly rotund (LUR) norms, gives

Theorem 1 (Asplund). If a Banach space \( X \) admits an equivalent LUR norm \( \| \cdot \|_1 \) and an equivalent norm \( \| \cdot \|_2 \) whose dual norm is LUR, then \( X \) admits an equivalent LUR norm \( \| \| \) whose dual norm is also LUR.

Recall that an LUR norm is one which satisfies \( \lim_j \| x_j - x \| = 0 \) whenever \( x_j, x \in X \) and \( \lim_j 2(\| x_j \|^2 + \| x \|^2) - \| x + x_j \|^2 = 0 \).

We give here a short proof of the following weaker version of Theorem 1:

Theorem 1' (Asplund). Under the same assumptions as in Theorem 1, \( X \) admits an equivalent norm \( \| \| \) which is LUR and Fréchet differentiable (on \( X \setminus \{0\} \)).

Proof of Theorem 1'. For \( n \geq 3 \) let \( \| f \|_n^* = (\| f \|_1^2 + n^{-1} \| f \|_2^2)^{1/2} \). Each \( \| \cdot \|_n^* \) is clearly an LUR equivalent norm on \( X^* \), dual to some norm \( \| \cdot \|_n \) on \( X \). Furthermore, \( \lim_n \| x \|_n = \| x \|_1 \) uniformly on bounded sets of \( X \). Since each \( \| \cdot \|_n^* \) is LUR, the norm \( \| \cdot \|_n \) is Fréchet differentiable (cf. e.g. [2]). Consider the norm \( \| x \|_n = (\sum_{i=1}^{n} 2^{-i} \| x \|_i^2)^{1/2} \); this is an equivalent norm on \( X \). Since the differentials \( (\| \cdot \|_n^2)' \) of \( \| \cdot \|_n^2 \) are uniformly bounded on bounded sets of \( X \), the norm \( \| \cdot \|_n \) is Gâteaux differentiable and the differential \( (\| \cdot \|_n^2)' \) is norm-norm continuous (as all \( (\| \cdot \|_i^2)' \) are such—see e.g. [2]). Thus \( \| \cdot \|_n \) is Fréchet differentiable. To see that \( \| \cdot \| \) is LUR, suppose \( x_j, x \in X \), and \( \lim_j 2(\| x_j \|^2 + \| x \|^2) - \| x + x_j \|^2 = 0 \). Then the same is true for any \( \| \cdot \|_n \) and since \( \{ x_j \} \) is then necessarily bounded and \( \lim_n \| x \|_n = 0 \).

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\[ \|x\|_1 \text{ uniformly on bounded sets, we have } \lim_j 2(\|x_j\|_1^2 + \|x\|_1^2) - \|x_j + x\|_1^2 = 0. \] So, by LUR of the norm \( \| \cdot \|_1 \), we have \( \lim_j \|x_j - x\|_1 = 0. \)

Remark. The above argument also works for other properties (like rotundity, uniform rotundity, etc.). In the case where there is exact duality between a differentiability and a rotundity notion (e.g. uniform rotundity and uniform Fréchet differentiability, or rotundity and Gâteaux differentiability in reflexive spaces), our proof gives the original Theorem 1.

REFERENCES


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