A GENERALIZATION OF A THEOREM OF S. PICCARD

WOLFGANG SANDER

The following theorem is due to S. Piccard [2, p. 30]:
"The difference of two second category Baire sets (see [1]) contains a nonempty open set".

For various generalizations of this result the reader is referred to [3] and [4], where he can also find some more references.

In this note we give a short proof of a generalization of Piccard's theorem.

Let $f: X \times X \to X$. We define $f_1: X \to X$ and $f^2: X \to X$ by $f_1(y) = f(x, y)$ and $f^2(x) = f(x, y)$ for all $x, y \in X$. Then $f$ is globally solvable, if $f$ is continuous and if there exist two continuous functions $\psi, \phi: X \times X \to X$ such that $f(x, y) = z$ is equivalent to $x = \psi(y, z)$ and $y = \phi(x, z)$ for all $x, y, z \in X$. It follows that $f_1, f^2, \psi, \phi$ are homeomorphisms.

If $X$ is a topological group and $f(x, y) = x \cdot y$, then $\psi(y, z)$ and $\phi(x, z)$ may be taken to be $z \cdot y^{-1}$ and $x^{-1} \cdot z$.

**Theorem (cf. [4, Satz 7]).** Let $X$ be a topological space and let $f: X \times X \to X$ be a globally solvable function. If $A, B \subset X$ are of second category and $A$ has the property of Baire, then $f(A \times B)$ contains a nonempty open set and $X$ is a Baire space.

**Proof.** By hypothesis there exists a nonempty open set $G$ such that $G - A$ is of first category. For any set $C \subset X$ let $D(C)$ denote the set of all points of $X$ where $C$ is of second category. Then $D(C) \neq \emptyset$ if and only if $C$ is of second category, $G \cap D(C) = G \cap D(G \cap C)$ and $D(G \cap C) = D(G \cap A \cap C)$ [1, pp. 83–85]. Thus

$$f(G \times D(B)) = \bigcup \{f^2 G: y \in D(B)\}$$

is a nonempty open set. Since $D\gamma = \gamma D$ for each homeomorphism $\gamma$ of $X$, $\emptyset \neq f(G \times D(B)) = \{z \in X: G \cap \psi^z D(B) \neq \emptyset\} = \{z \in X: G \cap D(\psi^z G) \neq \emptyset\} = \{z \in X: G \cap D(G \cap A \cap \psi^z B) \neq \emptyset\} \subset \{z \in X: A \cap \psi^z B \neq \emptyset\} = f(A \times B)$.

The global solvability of $f$ implies that $X$ is a homogeneous space. Since $D(X)$ is invariant under every homeomorphism of $X$, it follows that $D(X) = X$, that is, $X$ is a Baire space.

Received by the editors May 10, 1978 and, in revised form, June 30, 1978.
Key words and phrases. Baire space, Baire set, second category set.

© 1979 American Mathematical Society
0002-9939/79/0000-0078/$01.50

281
ACKNOWLEDGEMENT. The author wishes to thank the referee for comments and advice.

REFERENCES


Institut für Mathematik, Technische Universität Braunschweig, D 33 Braunschweig, Deutschland