PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 3, March 1979

EMBEDDING OF A LIE ALGEBRA INTO LIE-ADMISSIBLE ALGEBRAS

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ABSTRACT. Let A be a flexible Lie-admissible algebra over a field of characteristic $\neq 2, 3$. Let S be a finite-dimensional classical Lie subalgebra of A^- which is complemented by an ideal R of A^-. It is shown that S is a Lie algebra under the multiplication in A and is an ideal of A if and only if S contains a classical Cartan subalgebra H which is nil in A and such that $HH \subseteq S$ and $[H, R] = 0$. In this case, the multiplication between S and R is determined by linear functionals on R which vanish on $[R, R]$. If A is finite-dimensional and of characteristic 0 then this can be applied to give a condition that a Levi-factor S of A^- be embedded as an ideal into A and to determine the multiplication between S and the solvable radical of A^-.

1. Introduction. For an algebra A, denote by A^- the algebra with multiplication $[x, y] = xy - yx$ defined on the vector space A. If A^- is a Lie algebra then A is said to be Lie-admissible. If A satisfies the flexible law $(xy)x = x(yx)$ for all $x, y \in A$ then A is called a flexible algebra. A linearized form of the flexible law is $(xy)z - x(yz) + (zy)x - z(yx) = 0$. If A is flexible and Lie-admissible then it is well known that the mapping ad x: $a \to [a, x]$ is a derivation of A for all $x \in A$; that is, $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in A$. Recent results show that a Cartan subalgebra of A^- plays a major role for the structure of flexible Lie-admissible algebras A [1], [2], [3]. Possible applications of Lie-admissible algebras in physics have been recently pointed out by a number of physicists. For this, the reader is referred to Santilli's recent work [5].

The purpose of this paper is to give a condition in terms of a Cartan subalgebra that a classical Lie algebra S be embedded as an ideal into a flexible Lie-admissible algebra A when the subspace S is complemented by an ideal R of A^-, and then to determine the multiplication between S and R. We make use of the known structure for classical Lie algebras [4] and the result that if A is a flexible algebra and A^- is a classical Lie algebra with a classical Cartan subalgebra which is nil in A, then A is a Lie algebra isomorphic to A^- [2]. A subset M of A said to be nil in A if every element of M is power-associative and nilpotent in A.

Received by the editors March 27, 1978.

AMS (MOS) subject classifications (1970). Primary 17A20, 17B05.

Key words and phrases. Flexible algebra, Lie-admissible algebra, classical Lie algebra, Cartan subalgebra, Levi-factor.

Let S be the direct sum of Lie algebras $S_1, \ldots, S_n$ over a field Φ of characteristic $\neq 2$ and R be a flexible Lie-admissible algebra over Φ. Let $f_1, \ldots, f_n$ be linear functionals on R which vanish on $[R, R]$. Define a multiplication in the vector space direct sum $A = S + R$ as follows:

$$\left(\sum_{i=1}^{n} x_i + r\right)\left(\sum_{i=1}^{n} y_i + s\right) = \sum_{i=1}^{n} [x_i y_i + f_i(r)y_i + f_i(s)x_i] + rs,$$
$$x_i, y_i \in S_i, r, s \in R, i = 1, 2, \ldots, n. \quad (*)$$

Then one sees that $[x + r, y + s] = 2xy + [r, s]$ for $x, y \in S$ and $r, s \in R$ and so A is Lie-admissible. One also computes

$$[(x + r)(y + s)](x + r) - (x + r)[(y + s)(x + r)] = \sum_i f_i([r, s])x_i = 0,$$

where $x = \sum_i x_i$, since $f_i([R, R]) = 0$. Thus A is flexible. Clearly, R is a subalgebra of A and is an ideal of A^-. The Lie algebra S is embedded as an ideal in A and if $x \in S_i$ then $xr = rx = f_i(r)x$ for $r \in R$. In fact, we show that this is the essential source for the embedding in question. We state the main result as follows.

THEOREM 1. *Let A be a flexible Lie-admissible algebra over a field Φ of characteristic $\neq 2, 3$ (not necessarily finite-dimensional). Let S be a finite-dimensional classical subalgebra of A^- which is complemented by an ideal R of A^-. Then S is a Lie algebra under the multiplication in A and is an ideal of A if and only if S contains a classical Cartan subalgebra H which is nil in A and such that $HH \subseteq S$ and $[H, R] = 0$. In this case, R is a subalgebra of A and the multiplication in A is given by the rule $(*)$ where $f_1, \ldots, f_n$ are linear functionals on R which vanish on $[R, R]$, and n is the number of simple summands in S.*

Recall that a finite-dimensional classical Lie algebra is a direct sum of simple Lie algebras [4].

2. Proof of Theorem 1. We begin with the following lemma.

LEMMA. *Let L be a Lie algebra over an arbitrary field Φ. Let S be a finite-dimensional subalgebra of L and H be a Cartan subalgebra of S. Then, for an ideal R of L, $[SR] = 0$ if and only if $[HR] = 0$.*

PROOF. One may assume that Φ is algebraically closed; if not, one takes the scalar extension of S to the algebraic closure of Φ. Let $S = \sum_\alpha S_\alpha$ be the Cartan decomposition for S relative to H. For each nonzero root α, choose an $h \in H$ such that $\alpha(h) \neq 0$. Then ad h: $S_\alpha \to S_\alpha$ is surjective; for, if not then there is an element $x \neq 0$ in S_α such that $[x, h] = 0$ and this together with $x(\text{ad } h - \alpha(h)I)^n = 0$ implies that $\alpha(h)^n x = 0$. This is absurd. Thus we have that $[S_\alpha h] = S_\alpha$ for $\alpha \neq 0$ and $\alpha(h) \neq 0$. If $[HR] = 0$ and $\alpha \neq 0$ then, by the Jacobi identity, $[S_\alpha R] = [[S_\alpha h]R] \subseteq [[Rh]S_\alpha] + [[S_\alpha R]h] = 0$ since R is an ideal of L, and so $[SR] = 0$.

For the proof of Theorem 1, we first observe that the centralizer $C_{A^-}(M)$ of

a subset M of A in A^- is a subalgebra of A, since $[xy, M] \subseteq x[y, M] + [x, M]y = 0$ for all $x, y \in C(M)$. Suppose that S is a finite-dimensional classical subalgebra of A^- having a classical Cartan subalgebra H which is nil in A and that $HH \subseteq S$ and $[H, R] = 0$. That $HH \subseteq S$ implies that S is a subalgebra of A [3]. It then follows from [2, Corollary 3.4] that S is a Lie algebra under the multiplication in A. Thus we have that $[x, y] = xy - yx = 2xy$ for $x, y \in S$. By the Lemma, we get $[S, R] = 0$ and since S has center 0, $R = C_{A^-}(S)$ and so R is a subalgebra of A.

Let $x, y \in S$ and let $r \in R$. Write $yr = z + s$ for some $z \in S$ and $s \in R$. Then the flexible law $(xy)r - x(yr) + (ry)x - r(yx) = 0$ implies $(xy)r = xz$ and so $(SS)R \subseteq S$. Since $SS = S$, this proves that S is an ideal of A. Since $C_S(H) = H$ and $HH = [H, H] = 0$, from $[h', hr] = [h', h]r + h[h', r] = 0$ for all $h, h' \in H$ and $r \in R$, we have

$$HR = RH \subseteq H. \tag{1}$$

Let $S = \sum_\alpha S_\alpha$ be the Cartan decomposition for S relative to H. Note that $S_0 = H$ and $\dim S_\alpha = 1$ for $\alpha \neq 0$ since S is classical. Then $xh = \alpha(h)x$ for $x \in S_\alpha$ and $h \in H$. For each nonzero root α, choose an $h \in H$ with $\alpha(h) \neq 0$. If $x \in S_\alpha$ and $r \in R$ then, from $(xh)r - x(hr) + (rh)x - r(hx) = 0$ and (1), one gets $xr = \alpha(h)^{-1}\alpha(hr)x$ and so

$$xr = rx = \lambda x, x \in S_\alpha, \qquad \alpha \neq 0, \tag{2}$$

where $\lambda \in \Phi$ depends on $r \in R$ and $\alpha \neq 0$. Since $S_\alpha S_{-\alpha}$ is one-dimensional for $\alpha \neq 0$, one chooses nonzero elements $x \in S_\alpha, y \in S_{-\alpha}, h \in H$ such that

$$xh = x, \quad yh = -y, \quad xy = h. \tag{3}$$

Then by (2) we have that $xr = rx = \lambda x$ and $yr = ry = \mu y$ for $\mu \in \Phi$ and $r \in R$. If $H = \Phi x + B$ is a vector space direct sum then by (1) we can let $hr = rh = \nu h + b$, $\nu \in \Phi$, $b \in B$. From (3) and $(xy)r - x(yr) + (ry)x - r(yx) = 0$, one gets $2(\nu h + b) = 2\mu h$, so $b = 0$ and $\nu = \mu$, and by symmetry $\nu = \lambda$. Therefore

$$xr = rx = f_\alpha(r)x, \quad yr = ry = f_\alpha(r)y, \quad hr = rh = f_\alpha(r)h \tag{4}$$

for $\alpha \neq 0$ and $r \in R$, where f_α is a linear functional on R and $\{x, y, h\}$ is the canonical basis as in (3). In particular, we have that $f_\alpha = f_{-\alpha}$.

Recall that S is the direct sum of simple classical Lie algebras, so that each simple summand has a fundamental system of roots relative to a classical Cartan subalgebra which is connected. Thus we may assume that S is simple. If α, β ($\beta \neq 0$) is an ordered pair of roots then recall the Cartan integer $A_{\alpha,\beta} = r - q$ where r and q are the least nonnegative integers such that $\alpha - (r + 1)\beta$ and $\alpha + (q + 1)\beta$ are not roots. Let $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ be a fundamental system of roots which is connected; that is, for any two roots α, $\beta \in \Pi$, there are roots $\mu_1, \ldots, \mu_r \in \Pi$ such that $\alpha = \mu_1$, $\beta = \mu_r$ and $A_{\mu_i,\mu_{i+1}} \neq 0$, $1 \leqslant i \leqslant r$. For brevity, denote $A_{\alpha_i,\alpha_j} = A_{ij}$ and $f_{\alpha_i} = f_i$. We first show that if $A_{ij} \neq 0$ then $f_i = f_j$. If $A_{ij} < 0$ then $S_{\alpha_i} S_{\alpha_j} \neq 0$ and so choose elements

$x \in S_{\alpha_i}, y \in S_{\alpha_j}$ with $xy \neq 0$. Then the flexible law $(xy)r - x(yr) + x(yr) - r(yx) = 0$ with (4) implies that $(xy)r = f_j(r)(xy)$ for all $r \in R$. By symmetry we have that $(xy)r = f_i(r)(xy)$ and so $f_i = f_j$. If $A_{ij} > 0$ then, using the relation $A_{\alpha,-\beta} = -A_{\alpha,\beta}$ and $f_\alpha = f_{-\alpha}$, we argue that $f_i = f_j$. Since Π is connected, this proves that $f_i = f_j$ for all $i, j = 1, 2, \ldots, m$. Denote one of these linear functionals f_i by f.

Let x_i, y_i, h_i be the canonical basis corresponding to the root $\alpha_i \in \Pi$. We have then shown

$$x_i r = r x_i = f(r) x_i, \quad y_i r = r y_i = f(r) y_i, \quad h_i r = r h_i = f(r) h_i, \qquad r \in R, i = 1, 2, \ldots, m. \tag{5}$$

Since, for any root $\alpha \neq 0$, α or $-\alpha$ is a sum of roots in Π, there is a basis for S which consists of elements of the form

$$h_i, \quad (\cdots(x_{i_1} x_{i_2}) \cdots x_{i_k}), \quad (\cdots(y_{i_1} y_{i_2}) \cdots y_{i_k}), \qquad i = 1, 2, \ldots, m, \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, m\}.$$

The flexible law $(x_i x_j)r - x_i(x_j r) + (r x_j) x_i - r(x_j x_i) = 0$ gives $(x_i x_j) r = f(r)(x_i x_j)$ by (5). Therefore, by induction, we have

$$xr = rx = f(r)x, \qquad x \in S, r \in R.$$

Finally, let $r, s \in R$ and let x be a nonzero element of S. Then $(rs)x - r(sx) + (xs)r - x(sr) = 0$ gives $f([r, s]) = 0$ and so f vanishes on $[R, R]$. Thus the multiplication in A is given by $(*)$ and this completes the proof.

3. Application. Let A be a finite-dimensional flexible Lie-admissible algebra over a field of characteristic 0. Then, by Levi's theorem, every Levi-factor S (a maximal semisimple subalgebra of A^-) of A^- is complemented by the solvable radical of A^-. It then follows from Corollary of [3] that if S is power-associative in A and contains a Cartan subalgebra H with $HH \subseteq S$ then S is a subalgebra of A and is a Lie algebra under the multiplication in A. Therefore, in view of Theorem 1, we have

THEOREM 2. *Let A be a finite-dimensional flexible Lie-admissible algebra over a field of characteristic* 0. *Let R be the solvable radical of A^- and S be a Levi-factor of A^- which is power-associative in A. Then S is a Lie algebra under the multiplication in A and is an ideal of A if and only if S contains a split Cartan subalgebra H such that $HH \subseteq S$ and $[H, R] = 0$. In this case, the multiplication in A is given by* $(*)$.

Theorem 2 strengthens Theorem 4.1 in [2] which requires the additional assumptions that the radical R of A^- is nilpotent in A^- and H is nil in A. If, in Theorem 1, the complementary subalgebra R of A satisfies $[R, R] = R$, then the linear functionals f_i are 0. Thus we have

COROLLARY 1. *Let A, S, R be the same as in Theorem* 1 *and let S be embedded as in Theorem* 1. *If $[R, R] = R$ then R is an ideal of A. In particular, if R^- is a simple Lie algebra then R is an ideal of A.*

COROLLARY 2. *Let A, S, R be the same as in Corollary* 1 *and of characteristic* 0. *If R^- is nilpotent and R is nil in A then R is an ideal of A.*

If R^- is nilpotent then ad x is nilpotent in A for all $x \in R$. Since R is a nilalgebra, it follows from [2, Lemma 4.4] that the right multiplication $R(x)$ in A by $x \in R$ is nilpotent. Hence the linear functionals f_i are 0 and this proves Corollary 2.

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