EMBEDDING OF A LIE ALGEBRA INTO LIE-ADMISSIBLE ALGEBRAS

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Abstract. Let $A$ be a flexible Lie-admissible algebra over a field of characteristic $\neq 2, 3$. Let $S$ be a finite-dimensional classical Lie subalgebra of $A^-$ which is complemented by an ideal $R$ of $A^-$. It is shown that $S$ is a Lie algebra under the multiplication in $A$ and is an ideal of $A$ if and only if $S$ contains a classical Cartan subalgebra $H$ which is nil in $A$ and such that $HH \subseteq S$ and $[H, R] = 0$. In this case, the multiplication between $S$ and $R$ is determined by linear functionals on $R$ which vanish on $[R, R]$. If $A$ is finite-dimensional and of characteristic 0 then this can be applied to give a condition that a Levi-factor $S$ of $A^-$ be embedded as an ideal into $A$ and to determine the multiplication between $S$ and the solvable radical of $A^-$. 

1. Introduction. For an algebra $A$, denote by $A^-$ the algebra with multiplication $[x, y] = xy - yx$ defined on the vector space $A$. If $A^-$ is a Lie algebra then $A$ is said to be Lie-admissible. If $A$ satisfies the flexible law $(xy)x = x(yx)$ for all $x, y \in A$ then $A$ is called a flexible algebra. A linearized form of the flexible law is $(xy)z - x(yz) + (zy)x - z(yx) = 0$. If $A$ is flexible and Lie-admissible then it is well known that the mapping $\text{ad} x: a \mapsto [a, x]$ is a derivation of $A$ for all $x \in A$; that is, $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in A$. Recent results show that a Cartan subalgebra of $A^-$ plays a major role for the structure of flexible Lie-admissible algebras $A$ [1], [2], [3]. Possible applications of Lie-admissible algebras in physics have been recently pointed out by a number of physicists. For this, the reader is referred to Santilli's recent work [5].

The purpose of this paper is to give a condition in terms of a Cartan subalgebra that a classical Lie algebra $S$ be embedded as an ideal into a flexible Lie-admissible algebra $A$ when the subspace $S$ is complemented by an ideal $R$ of $A^-$, and then to determine the multiplication between $S$ and $R$. We make use of the known structure for classical Lie algebras [4] and the result that if $A$ is a flexible algebra and $A^-$ is a classical Lie algebra with a classical Cartan subalgebra which is nil in $A$, then $A$ is a Lie algebra isomorphic to $A^-$ [2]. A subset $M$ of $A$ said to be nil in $A$ if every element of $M$ is power-associative and nilpotent in $A$. 

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Let $S$ be the direct sum of Lie algebras $S_1, \ldots, S_n$ over a field $\Phi$ of characteristic $\neq 2$ and $R$ be a flexible Lie-admissible algebra over $\Phi$. Let $f_1, \ldots, f_n$ be linear functionals on $R$ which vanish on $[R, R]$. Define a multiplication in the vector space direct sum $A = S + R$ as follows:

$$
(\sum_{i=1}^{n} x_i + r) (\sum_{i=1}^{n} y_i + s) = \sum_{i=1}^{n} [x_i y_i + f_i(r) y_i + f_i(s) x_i] + rs,
$$

where $x_i, y_i \in S_i, r, s \in R, i = 1, 2, \ldots, n.$ (*)

Then one sees that $[x + r, y + s] = 2xy + [r, s]$ for $x, y \in S$ and $r, s \in R$ and so $A$ is Lie-admissible. One also computes

$$
[(x + r)(y + s)](x + r) - (x + r)[(y + s)(x + r)] = \sum_i f_i([r, s]) x_i = 0,
$$

where $x = \sum_i x_i$, since $f_i([R, R]) = 0$. Thus $A$ is flexible. Clearly, $R$ is a subalgebra of $A$ and is an ideal of $A^-$. The Lie algebra $S$ is embedded as an ideal in $A$ and if $x \in S$, then $xr = rx = f_i(r)x$ for $r \in R$. In fact, we show that this is the essential source for the embedding in question. We state the main result as follows.

**Theorem 1.** Let $A$ be a flexible Lie-admissible algebra over a field $\Phi$ of characteristic $\neq 2, 3$ (not necessarily finite-dimensional). Let $S$ be a finite-dimensional classical subalgebra of $A^-$ which is complemented by an ideal $R$ of $A^-$. Then $S$ is a Lie algebra under the multiplication in $A$ and is an ideal of $A$ if and only if $S$ contains a classical Cartan subalgebra $H$ which is nil in $A$ and such that $HH \subseteq S$ and $[H, R] = 0$. In this case, $R$ is a subalgebra of $A$ and the multiplication in $A$ is given by the rule (*) where $f_1, \ldots, f_n$ are linear functionals on $R$ which vanish on $[R, R]$, and $n$ is the number of simple summands in $S$.

Recall that a finite-dimensional classical Lie algebra is a direct sum of simple Lie algebras $[4]$.

2. Proof of Theorem 1. We begin with the following lemma.

**Lemma.** Let $L$ be a Lie algebra over an arbitrary field $\Phi$. Let $S$ be a finite-dimensional subalgebra of $L$ and $H$ be a Cartan subalgebra of $S$. Then, for an ideal $R$ of $L$, $[SR] = 0$ if and only if $[HR] = 0$.

**Proof.** One may assume that $\Phi$ is algebraically closed; if not, one takes the scalar extension of $S$ to the algebraic closure of $\Phi$. Let $S = \sum_a S_a$ be the Cartan decomposition for $S$ relative to $H$. For each nonzero root $\alpha$, choose an $h \in H$ such that $\alpha(h) \neq 0$. Then $\text{ad} ~ h: S_a \to S_a$ is surjective; for, if not then there is an element $x \neq 0$ in $S_a$ such that $[x, h] = 0$ and this together with $x(\text{ad} ~ h - \alpha(h)I)x = 0$ implies that $\alpha(h)x = 0$. This is absurd. Thus we have that $[S_a h] = S_a$ for $a \neq 0$ and $\alpha(h) \neq 0$. If $[HR] = 0$ and $\alpha \neq 0$ then, by the Jacobi identity, $[S_a R] = [[S_a h] R] \subseteq [[Rh] S_a] + [[S_a R] h] = 0$ since $R$ is an ideal of $L$, and so $[SR] = 0$.

For the proof of Theorem 1, we first observe that the centralizer $C_A^-(M)$ of
a subset $M$ of $A$ in $A^-$ is a subalgebra of $A$, since $[xy, M] \subseteq x[y, M] + [x, M]y = 0$ for all $x, y \in C(M)$. Suppose that $S$ is a finite-dimensional classical subalgebra of $A^-$ having a classical Cartan subalgebra $H$ which is nil in $A$ and that $HH \subseteq S$ and $[H, R] = 0$. That $HH \subseteq S$ implies that $S$ is a subalgebra of $A$ [3]. It then follows from [2, Corollary 3.4] that $S$ is a Lie algebra under the multiplication in $A$. Thus we have that $[x, y] = xy - yx = 2xy$ for $x, y \in S$. By the Lemma, we get $[S, R] = 0$ and since $S$ has center 0, $R = C_A(S)$ and so $R$ is a subalgebra of $A$.

Let $x, y \in S$ and let $r \in R$. Write $yr = z + s$ for some $z \in S$ and $s \in R$. Then the flexible law $(xy)r - x(yr) + (ry)x - r(yx) = 0$ implies $(xy)r = xz$ and so $(SS)R \subseteq S$. Since $SS = S$, this proves that $S$ is an ideal of $A$. Since $C_S(H) = H$ and $HH = [H, H] = 0$, from $[h', hr] = [h', h]r + h[h', r] = 0$ for all $h, h' \in H$ and $r \in R$, we have

$$HH = RH \subseteq H.$$  \hfill (1)

Let $S = \sum_S S_a$ be the Cartan decomposition for $S$ relative to $H$. Note that $S_0 = H$ and $\dim S_a = 1$ for $a \neq 0$ since $S$ is classical. Then $xh = \alpha(h)x$ for $x \in S_a$ and $h \in H$. For each nonzero root $a$, choose an $h \in H$ with $\alpha(h) \neq 0$. If $x \in S_a$ and $r \in R$ then, from $(xh) - x(hr) + (rh)x - r(hx) = 0$ and \hfill (1), one gets $xr = \alpha(h)x$ and so

$$xr = rx = \lambda x, \quad x \in S_a, \quad \alpha \neq 0,$$  \hfill (2)

where $\lambda \in \Phi$ depends on $r \in R$ and $\alpha \neq 0$. Since $S_a S_{-a}$ is one-dimensional for $a \neq 0$, one chooses nonzero elements $x \in S_a, y \in S_{-a}, h \in H$ such that

$$xh = x, \quad yh = -y, \quad xy = h.$$  \hfill (3)

Then by (2) we have that $xr = rx = \lambda x$ and $yr = ry = \mu y$ for $\mu \in \Phi$ and $r \in R$. If $H = \Phi x + B$ is a vector space direct sum then by (1) we can let $hr = rh = vh + b, \quad v \in \Phi, \quad b \in B$. From (3) and $(xy)r - x(yr) + (ry)x - r(yx) = 0$, one gets $2(vh + b) = 2vh$, so $b = 0$ and $v = \mu$, and by symmetry $v = \lambda$. Therefore

$$xr = rx = f_\alpha(r)x, \quad yr = ry = f_\alpha(r)y, \quad hr = rh = f_\alpha(r)h$$  \hfill (4)

for $a \neq 0$ and $r \in R$, where $f_\alpha$ is a linear functional on $R$ and $\{x, y, h\}$ is the canonical basis as in (3). In particular, we have that $f_\alpha = f_{-\alpha}$.

Recall that $S$ is the direct sum of simple classical Lie algebras, so that each simple summand has a fundamental system of roots relative to a classical Cartan subalgebra which is connected. Thus we may assume that $S$ is simple. If $a, \beta$ ($\beta \neq 0$) is an ordered pair of roots then recall the Cartan integer $A_{a, \beta} = r - q$ where $r$ and $q$ are the least nonnegative integers such that $a - (r + 1)\beta$ and $\alpha + (q + 1)\beta$ are not roots. Let $\Pi = \{a_1, \ldots, a_m\}$ be a fundamental system of roots which is connected; that is, for any two roots $a, \beta \in \Pi$, there are roots $\mu_1, \ldots, \mu_r \in \Pi$ such that $a = \mu_1, \beta = \mu_r$ and $A_{\mu_i, \mu_{i+1}} \neq 0, 1 < i < r$. For brevity, denote $A_{a, a} = A_{ij}$ and $f_{a} = f_i$. We first show that if $A_{ij} \neq 0$ then $f_i = f_j$. If $A_{ij} < 0$ then $S_{a_1} S_{a_2} \neq 0$ and so choose elements
Let $x_i, y_i, h_i$ be the canonical basis corresponding to the root $\alpha_i \in \Pi$. We have then shown

$$x_i r = r x_i = f(r) x_i, \quad y_i r = r y_i = f(r) y_i, \quad h_i r = r h_i = f(r) h_i,$$

$$r \in R, \quad i = 1, 2, \ldots, m. \quad (5)$$

Since, for any root $\alpha \neq 0$, $\alpha$ or $-\alpha$ is a sum of roots in $\Pi$, there is a basis for $S$ which consists of elements of the form

$$h_i, \quad \left( \cdots (x_{i_1} x_{i_2}) \cdots x_{i_k} \right), \quad \left( \cdots (y_{i_1} y_{i_2}) \cdots y_{i_k} \right),$$

$$i = 1, 2, \ldots, m, \quad \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, m\}.$$

The flexible law $(x_i x_j) r = x_i (x_j r) + (r x_j) x_i - r (x_i x_j) = 0$ gives $(x_i x_j) r = f(r) (x_i x_j)$ by (5). Therefore, by induction, we have

$$x r = r x = f(r) x, \quad x \in S, r \in R.$$

Finally, let $r, s \in R$ and let $x$ be a nonzero element of $S$. Then $(rs)x - r(xs) + (xs)r - x(sr) = 0$ gives $f([r, s]) = 0$ and so $f$ vanishes on $[R, R]$. Thus the multiplication in $A$ is given by $(\ast)$ and this completes the proof.

3. Application. Let $A$ be a finite-dimensional flexible Lie-admissible algebra over a field of characteristic 0. Then, by Levi's theorem, every Levi-factor $S$ (a maximal semisimple subalgebra of $A^-$) of $A^-$ is complemented by the solvable radical of $A^-$. It then follows from Corollary of [3] that if $S$ is power-associative in $A$ and contains a Cartan subalgebra $H$ with $HH \subseteq S$ then $S$ is a subalgebra of $A$ and is a Lie algebra under the multiplication in $A$. Therefore, in view of Theorem 1, we have

**Theorem 2.** Let $A$ be a finite-dimensional flexible Lie-admissible algebra over a field of characteristic 0. Let $R$ be the solvable radical of $A^-$ and $S$ be a Levi-factor of $A^-$ which is power-associative in $A$. Then $S$ is a Lie algebra under the multiplication in $A$ and is an ideal of $A$ if and only if $S$ contains a split Cartan subalgebra $H$ such that $HH \subseteq S$ and $[H, R] = 0$. In this case, the multiplication in $A$ is given by $(\ast)$.

Theorem 2 strengthens Theorem 4.1 in [2] which requires the additional assumptions that the radical $R$ of $A^-$ is nilpotent in $A^-$ and $H$ is nil in $A$. If, in Theorem 1, the complementary subalgebra $R$ of $A$ satisfies $[R, R] = R$, then the linear functionals $f_i$ are 0. Thus we have

**Corollary 1.** Let $A, S, R$ be the same as in Theorem 1 and let $S$ be embedded as in Theorem 1. If $[R, R] = R$ then $R$ is an ideal of $A$. In particular, if $R^-$ is a simple Lie algebra then $R$ is an ideal of $A$. 

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Corollary 2. Let $A$, $S$, $R$ be the same as in Corollary 1 and of characteristic 0. If $R^-$ is nilpotent and $R$ is nil in $A$ then $R$ is an ideal of $A$.

If $R^-$ is nilpotent then $\text{ad } x$ is nilpotent in $A$ for all $x \in R$. Since $R$ is a nilalgebra, it follows from [2, Lemma 4.4] that the right multiplication $R(x)$ in $A$ by $x \in R$ is nilpotent. Hence the linear functionals $f_i$ are 0 and this proves Corollary 2.

References


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