A RELAXED PICARD ITERATION PROCESS
FOR SET-VALUED OPERATORS OF THE MONOTONE TYPE

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Abstract. The fixed points $\bar{x}$ of set-valued operators, $T: X \to 2^X$, satisfying a condition of the monotonicity type on convex subsets $X$ of a Hilbert space are approximated by a relaxation process, $x_{n+1} = x_n + \omega_n(Tx_n - x_n)$, in which $T$ is a single-valued branch of $T$ and the relaxation parameter $\omega_n \in [0, 1]$ is made to depend in a certain way on the prior history of the process. If $T$ is bounded on bounded subsets of $X$, then $\|x_n - \bar{x}\|$ converges to 0 like $O(n^{-1/2})$. If $T$ is also continuous at $\bar{x}$ and if $\bar{x} = Tx$, then $\|x_n - \bar{x}\| = o(n^{-1/2})$. If $T$ satisfies a condition of the Lipschitz type at $\bar{x}$, then $\|x_n - \bar{x}\| = O(\mu n^{2})$ for some $\mu \in [0, 1)$.

1. Introduction. Let $X$ be a nonempty convex subset of a Hilbert space $H$, and suppose that for some $\bar{x} \in X$, the set-valued map $T: X \to 2^X$ satisfies the monotonicity condition,

$$1 > C_* \Delta \sup_{x \in X} \sup_{x \neq \bar{x}} \frac{\text{Re} \langle \xi - \bar{x}, x - \bar{x} \rangle}{\|X - \bar{x}\|^2};$$

where

$$(1.1)$$

(every fixed point of $T$ necessarily coincides with $\bar{x}$). Suppose also that the range of $T$ is bounded in the sense that

$$\text{diam } \bigcup_{x \in X} Tx < \infty$$

and therefore,

$$\sup_{x \in X} \sup_{\xi \in T_x} \|\xi - \bar{x}\| < \infty.$$

If $\tilde{T}$ is any single-valued branch of $T$ and if $(x_n) \subset X$ is generated by the associated relaxation process,

$$x_{n+1} = x_n + \omega_n(\tilde{T}x_n - x_n), \quad x_0 \in X,$$

where the relaxation parameter sequence $(\omega_n) \subset [0, 1]$ satisfies

$$\sum_{n=0}^{\infty} \omega_n^2 < \infty$$

$$(1.5A)$$

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and

$$\sum_{n=0}^{\infty} \omega_n = \infty.$$  \hspace{1cm} (1.5B)

Then \( \{x_n\} \) converges strongly to \( \bar{x} \). Moreover, for certain special parameter sequences of this type, one can show that \( \|x_n - \bar{x}\| = O(n^{-1/2}) \) [1], [2], [3] (for related results see [4] also). To construct such a sequence, it is enough to have a value of \( C \) satisfying \( 1 > C > C_* \); no additional information about \( T \) or the course of the iteration (1.4) is required, hence the parameter sequence \( \{\omega_n\} \) can be computed before (1.4) commences.

Roughly speaking, condition (1.5) requires that \( \{\omega_n\} \) converge to 0 but not “too quickly”. This restriction is necessary if (1.4) is to work in the worst cases where \( \tilde{T} \) is discontinuous at \( \bar{x} \). Without continuity, it is possible for the residual norms \( \|\tilde{T}x_n - x_n\| \) to remain bounded away from zero while \( x_n \) converges to a fixed point \( \bar{x} \) of \( \tilde{T} \), and this behavior is generally incompatible with (1.4) unless \( \omega_n \to 0 \). Unfortunately, while (1.5) insures convergence in the worst cases, it virtually guarantees a uniformly poor asymptotic rate of convergence in all cases, even where \( \tilde{T} \) is Lipschitz continuous. On the other hand, it is known that for Lipschitz continuous \( \tilde{T} \) satisfying (1.1), the sequences \( \{x_n\} \subset X \) generated by (1.4) will converge geometrically, or linearly, to \( \bar{x} \) (i.e., \( \|x_n - \bar{x}\| = O(\lambda^n) \) for some \( \lambda \in [0, 1) \)), provided the terms in \( \{\omega_n\} \) are sufficiently small but at the same time bounded away from 0 [5]–[9]. In fact, let \( \tilde{T} \) satisfy the condition

$$K_* = \Delta \sup_{x \in X} \frac{\|\tilde{T}x - \bar{x}\|}{\|x - \bar{x}\|} < \infty.$$  \hspace{1cm} (1.6)

For \( \omega_n \in [0, 1] \), (1.1) and (1.4) then give

$$\|x_{n+1} - \bar{x}\|^2 \leq \lambda_* (\omega_n) \|x_n - \bar{x}\|^2 \leq \lambda(\omega_n) \|x_n - \bar{x}\|^2$$  \hspace{1cm} (1.7A)

with

$$\lambda_* (\omega) = 1 - 2(1 - C_*)\omega + (1 - 2C_* + K_*^2)\omega^2$$  \hspace{1cm} (1.7B)

and

$$\lambda(\omega) = 1 - 2(1 - C)\omega + (1 - 2C + K^2)\omega^2$$  \hspace{1cm} (1.7C)

where

$$1 > C > C_*$$  \hspace{1cm} (1.7D)

and

$$K > K_*.$$  \hspace{1cm} (1.7E)

If for all \( n > 0 \), \( \omega_n = \omega \) with \( \omega \) positive and sufficiently small, the parameter \( \lambda(\omega_n) \) (and a fortiori \( \lambda_* (\omega_n) \)) will be nonnegative and smaller than 1, hence \( \{x_n\} \) will converge to \( \bar{x} \) geometrically. Furthermore for any given bounds
$C > C_*$ and $K > K_*$, there is a "best" value $\omega = \omega_{\text{OPT}}$ which minimizes the parameter $\lambda(\omega)$ in the bound (1.7A). To see this, observe first that for fixed $K$ and fixed $\omega \in [0, 1]$, $\lambda(\omega)$ increases with increasing $C$. Furthermore, since the Schwarz inequality, (1.1) and (1.6) give $|C_\omega| < K_\omega$, it follows that $K > K_\omega$ is always a better estimate of $C_\omega$ than any $C > K$. Therefore, consideration may be limited to $C$'s in the range,

$$|C| < K \quad \text{and} \quad C < 1. \quad (1.7F)$$

In this event, the coefficient of $\omega^2$ in (1.7C) is always positive and $\lambda(\omega)$ attains its least value over $\omega \in [0, 1]$ at

$$\omega_{\text{OPT}} = \min\left\{1, \frac{1 - C}{1 - 2C + K^2}\right\} \quad (1.8A)$$

where

$$\lambda_{\text{OPT}} = \lambda(\omega_{\text{OPT}}) = \begin{cases} K^2, & \text{if } (1 - C) > 1 - 2C + K^2, \\ \frac{(K^2 - C^2)}{(1 - C)^2 + (K^2 - C^2)}, & \text{if } (1 - C) < 1 - 2C + K^2. \end{cases} \quad (1.8B)$$

In either case,

$$0 < \lambda_{*_{\text{OPT}}} < \lambda_{\text{OPT}} < 1 \quad (1.8C)$$

where $\omega_{*_{\text{OPT}}}$ and $\lambda_{*_{\text{OPT}}}$ are obtained by putting $C = C^*$ and $K = K^*$ in (1.8A) and (1.8B). These estimates show that asymptotic convergence rates far better than $O(n^{-1/2})$ are attainable with (1.4) in principle when $\tilde{T}$ is sufficiently continuous at $\bar{x}$. However, in order to apply the relaxation parameter rule (1.8), one must have values for $C$ and $K$ which satisfy (1.7D) and (1.7E); if (1.8) is used with a $K < K_*$ or $C \notin [C_*, 1)$, then $\tilde{T}_{\omega_{\text{OPT}}} = I + \omega_{\text{OPT}}(\tilde{T} - I)$ is not necessarily contractive and the corresponding sequence $\{x_n\}$ generated by (1.4) with $\omega_n = \omega_{\text{OPT}}$ may not converge to any limit.

The present article considers a family of iterative processes (1.4) in which $\omega_*$ is made to depend in a simply way on the prior history of $x_k$, for $0 \leq k \leq n$. If $T$ is set-valued and satisfies the monotonicity condition (1.1), the corresponding error norms $\|x_n - \bar{x}\|$ are of order $O(n^{-1/2})$ as $n \to \infty$ for any branch $\tilde{T}$ that is bounded on bounded subsets of $X$. $T$ need not satisfy the stronger boundedness condition (1.2) invoked in [I] and need not be continuous for this estimate to apply, however if $\tilde{T}$ is continuous at $\bar{x}$ and if $\bar{x} = \tilde{T}\bar{x}$, it turns out that $\|x_n - \bar{x}\| = o(n^{-1/2})$. Moreover, if $\tilde{T}$ satisfies condition (1.6) at $\bar{x}$, then the error norms converge to 0 geometrically, even though neither the constant $K_*$ nor an upper bound $K$ for $K_*$ actually appears in the rule which determines $\omega_n$ (and is therefore not required for an implementation of this rule).

In [10], Williamson considers a relaxation process (1.4) for $T: X \to 2^H$, with
\(\omega_n\) depending on \(x_n\) according to a rule which turns out to be essentially a special case of the scheme proposed here; strong convergence of \(x_n\) to \(\bar{x}\) is established when \(X\) is dense in \(H\) (or in a solid sphere \(\subset H\)), and when \(T\) is bounded on bounded subsets of \(X\). Useful a posteriori error estimates are also derived in [10], along with the basic \(O(n^{-1/2})\) asymptotic convergence rate estimate.

The principal analytical tools employed in the next section were developed in [11] for a different purpose.

2. Results. Put

\[ R_n = \bar{T}x_n - x_n \quad \text{and} \quad E_n = x_n - \bar{x}. \]

Then (1.4) yields

\[ E^m_{n+1} = E^m_n + \omega_n R_n \]

and it follows easily from the monotonicity condition (1.1) that for all \(n > 0\) and \(\omega_n > 0\),

\[ \frac{\|E^m_{n+1}\|^2}{\theta^2} < \left[ 1 - 2(1 - C)\omega_n \right] \frac{\|E^m_n\|^2}{\theta^2} + \frac{\omega_n^2}{\theta^2} \|R_n\|^2 \]

where \(\theta^2 \neq 0\) is a scaling parameter about which more will be said later, and where \(1 > C > C_\ast\).

Let

\[ \bar{\omega} = \min \left\{ 1, \frac{1}{2(1 - C)} \right\} > 0 \]

and suppose that

\[ 0 < \omega_n < \bar{\omega} < 1 \]

for all \(n > 0\). Then the iterates \(\{x_n\}\) generated by (1.4) cannot leave the convex set \(X\). Moreover, the bracketed term on the right side of (2.1) is never negative and consequently a simple induction yields

\[ 0 < \|E^m_n\|^2 < B\beta_n \]

for all \(n > 0\), where,

\[ B = \max \{ \theta^2, \|E_0\| \} \]

and the sequence \(\{\beta_n\} \subset [0, \infty)\) is recursively generated by

\[ \beta_{n+1} = \left[ 1 - 2(1 - C)\omega_n \right] \beta_n + \frac{\omega_n^2}{\theta^2} \|R_n\|^2, \quad \beta_0 = 1. \]

Since the error estimates (2.3) hold for \textit{any} step length sequence \(\{\omega_n\}\).
satisfying (2.2), they are valid when \( \omega_n \) is determined by the rule,

\[
\omega_n = \tilde{\omega}(x_n, \beta_n; \theta) = \begin{cases} 
0, & \text{if } R_n = 0, \\
\min \left\{ \tilde{\omega}, \frac{\theta^2(1 - C) \beta_n}{\|R_n\|^2} \right\}, & \text{if } R_n \neq 0.
\end{cases}
\]  

(2.4)

This scheme insures that \( \{x_n\} \) will terminate at \( x^* \) if \( x^* \) is a fixed point of \( f \), and otherwise minimizes the right side of (2.3C) over \( \omega_n \in [0, \tilde{\omega}] \), given \( \beta_n \) and \( R_n \).

**Lemma 2.1.** If the sequences \( \{\omega_n\} \), \( \{\beta_n\} \), and \( \{\|R_n\|\} \) satisfy (2.3C) and (2.4) for \( n > 0 \), then \( \{\beta_n\} \) is nonnegative and monotone nonincreasing.

**Proof.** It has already been shown that \( \beta_n > 0 \) for all \( n > 0 \). When \( \omega_n = 0 \) in (2.4), (2.3C) \( \Rightarrow \beta_{n+1} = \beta_n \); when \( \omega_n = \theta^2(1 - C) \beta_n/\|R_n\|^2 \),

\[
(2.3C) \Rightarrow \beta_{n+1} = \left[ 1 - \frac{\theta^2(1 - C)^2 \beta_n}{\|R_n\|^2} \right] \beta_n < \beta_n;
\]

finally, when \( \omega_n = \tilde{\omega} \),

\[
(2.3C) \Rightarrow \beta_{n+1} = \left[ 1 - 2(1 - C)\tilde{\omega} \right] \beta_n + \tilde{\omega}^2 \frac{\|R_n\|^2}{\theta^2} \left[ 1 - (1 - C)\tilde{\omega} \right] \beta_n < \max \left\{ \frac{C}{2}, \beta_n \right\} \beta_n < \beta_n.
\]

In all cases, \( \beta_{n+1} < \beta_n \). Q.E.D.

**Corollary 1.** Let \( T : X \to 2^X \) satisfy the monotonicity condition (1.1), let the single-valued branch \( \bar{T} \) in (1.4) be bounded on bounded subsets of \( X \), and let \( \{x_n\} \subset X \) be generated by (1.4)–(2.4). Then \( \{x_n\} \) and the associated residual sequence \( \{R_n\} \) are bounded.

**Proof.** \( \{\beta_n\} \) is monotone nonincreasing and (2.3A) and (2.3B) hold, therefore \( \|x_n - \bar{x}\| < B^{1/2} \) for all \( n > 0 \). Since \( T \) is bounded on bounded subsets, the sequence \( \{T x_n - \bar{x}\} \) is also bounded. Boundedness of \( \{T x_n - x_n\} \) is now immediate from the triangle inequality. Q.E.D.

**Lemma 2.2.** Let \( \{\beta_n\} \subset (0, \infty) \) and \( \{q_n\} \subset [0, \infty) \) satisfy

\[
\beta_{n+1} < \beta_n - q_n \beta_n^2; \quad \beta_0 = 1 \tag{2.5}
\]

for all \( n > 0 \). If \( q_n > q > 0 \) for \( n > 0 \), then

\[
0 < \beta_n < \frac{1}{1 + qn} = O(n^{-1}). \tag{2.6}
\]

Furthermore, if \( \lim_{n \to \infty} q_n = \infty \), then \( \beta_n = o(n^{-1}) \).
Proof. Put $\delta_k = 1/\beta_k$. Then for all $k > 0$, one has $\delta_k - q_k > 0$ and $\delta_{k+1} - \delta_k > q_k\delta_k/(\delta_k - q_k) > q_k$. Consequently, for all $n > 0$,

$$
\delta_n = \delta_0 + \sum_{k=0}^{n-1} (\delta_{k+1} - \delta_k) > 1 + \sum_{k=0}^{n-1} q_k.
$$

If $q_k > q$ for all $k$, one then obtains (2.6). If $q_n \to \infty$, then for any $M > 0$ there is an $L$ so large that $k > L \Rightarrow q_k > M$. For $n > L$ one then has

$$
0 < \frac{n}{\delta_n} < \frac{n}{1 + \sum_{k=0}^{L-1} q_k + \sum_{k=L}^{n-1} q_k} \leq \frac{n}{(n - L)M}
$$

and therefore $0 < \lim_{n \to \infty} n\beta_n < \lim_{n \to \infty} n\beta_n < 1/M$. Since $M$ can be arbitrarily large, it follows that $\lim_{n \to \infty} n\beta_n = 0$. Q.E.D.

Theorem 2.1. Let $X$ be a nonempty convex subset of a Hilbert space $H$, let $T: X \to 2^X$ satisfy (1.1), and let $\hat{T}$ be a single-valued branch of $T$ that is bounded on bounded subsets of $X$. Suppose that $\{x_n\} \subset X$, $\{\beta_n\} \subset [0, \infty)$, and $\{\omega_n\} \subset [0, \overline{\omega}]$ are generated by (1.4), (2.3C), and (2.4) with any $\theta \neq 0$, $C \in [C^*, 1)$, and $\overline{\omega} = \min\{1, 1/2(1 - C)\}$. Let $\{R_n\}$ and $\{E_n\}$ denote the associated residual sequence and error sequence in (2.1). Then $\{R_n\}$ is bounded and for all $n > 0$,

$$
0 < \|E_n\|^2 < \max\{\theta^2, \|E_0\|^2\} \cdot \beta_n
$$

with $\{\beta_n\}$ nonnegative and monotone nonincreasing. Furthermore, either there is an $N > 0$ such that

$$
x_n = \bar{x}, \quad \forall n > N
$$

or else

$$
0 < \beta_n < \frac{1}{1 + qn}, \quad \forall n > 0
$$

with

$$
q = \min\left\{A, \frac{\theta^2(1 - C)^2}{d^2}\right\}
$$

and

$$
\infty > d = d_* = \sup_{n \geq 0} \|R_n\|
$$

in which case $\|E_n\| = O(n^{-1/2})$. Finally, if $\bar{x}$ is a fixed point of $\hat{T}$ and if $\hat{T}$ is continuous at $\bar{x}$, then $\|x_n - \bar{x}\| = o(n^{-1/2}).$

This result is established in [7]; for convenience, the proof is repeated here.
Proof. Corollary 1 of Lemma 2.1 \( \Rightarrow \) \( \{R_n\} \) is bounded. (2.7) has already been established in (2.3). Lemma 2.1 \( \Rightarrow \) \( \{\beta_n\} \) nonnegative and monotone nonincreasing. If \( R_n = 0 \), then \( x_n \) is a fixed point of \( \tilde{T} \) and must coincide with \( \bar{x} \) because of (1.1) [1]. Furthermore, (1.4)–(2.4) \( \Rightarrow \omega_n = 0 \) and \( x_{n+1} = x_n = \bar{x} \). (2.8) now follows by induction.

If \( R_n \neq 0 \) and \( \beta_n > 0 \), then (2.3C) and (2.4) give

\[
0 < \beta_{n+1} < (1 - q_n \beta_n) \beta_n \tag{2.10A}
\]

with

\[
q_n = \min \left\{ \frac{A}{\beta_n}, \frac{\theta^2 (1 - C)^2}{\|R_n\|^2} \right\} > q. \tag{2.10B}
\]

Since \( \beta = 1 \), one obtains \( \beta_n > 0 \) for all \( n > 0 \) by induction if \( R_n \neq 0 \) for all \( n > 0 \). The estimate (2.9) is now immediate from Lemma 2.2.

Finally if \( \bar{x} = \tilde{T}\bar{x} \), if \( \tilde{T} \) is continuous at \( \bar{x} \), and if \( R_n \neq 0 \) for all \( n > 0 \), then (2.7) and (2.9) \( \Rightarrow \|R_n\| \rightarrow 0 \) and \( \beta_n \rightarrow 0 \) through positive values, in which case (2.10B) \( \Rightarrow \lim_{n \rightarrow \infty} q_n = \infty \). It now follows from (2.7), (2.10A) and Lemma 2.2 that \( \|E_n\| = O(n^{-1/2}) \). Q.E.D.

Theorem 2.2. Let the hypothesis of Theorem 2.1 hold and suppose in addition that \( \tilde{T} \) satisfies (1.6). If \( R_n \neq 0 \) for \( 0 < n < N \), and \( R_N = 0 \), then (2.8) holds, and for \( 0 < n < N \),

\[
0 < \beta_n < \mu^n, \tag{2.11A}
\]

with

\[
\mu = 1 - \min \left\{ A, \frac{\theta^2 (1 - C)}{B(K + 1)^2} \right\} < 1 - \min \left\{ A, \frac{\theta^2 (1 - C)}{B(K + 1)^2} \right\} < 1, \tag{2.11B}
\]

\[
A = 1 - \max \left\{ \frac{1}{2}, C \right\}, \tag{2.11C}
\]

\[
B = \max \left\{ \theta^2, \|E_0\|^2 \right\}, \tag{2.11D}
\]

and

\[
K > K_* \tag{2.11E}
\]

If \( R_n \neq 0 \) for all \( n > 0 \), then (2.11) holds for all \( n > 0 \) and it follows from (2.7) that \( \|E_n\| = O(\mu^n) \).

Proof. If \( R_n \neq 0 \) for \( 0 < n < N \), then \( \beta_n \) is positive and satisfies (2.10) for \( 0 < n < N \). Furthermore,

\[
(1.6) \Rightarrow \|R_n\|^2 = \|Tx_n - x_n\|^2
\leq \left( \|Tx_n - \bar{x}\| + \|x_n - \bar{x}\| \right)^2 \leq (K + 1)^2 \|E_n\|^2
\]
and therefore (2.7) and (2.10) give
\[
\frac{\theta^2 (1 - C)^2 \beta_n}{\|R_n\|^2} > \frac{\theta^2 (1 - C)^2}{B(K + 1)^2}
\]
and
\[
0 < 1 - q_n \beta_n < \mu
\]
for \(0 < n < N\). Q.E.D.

Note 2.1. Theorems 2.1 and 2.2 hold for any nonzero value of the scaling parameter \(\theta\). If \(E\) is any upper bound on \(\|E_0\|\), then (2.7) and (2.8) give the estimate
\[
\|E_n\|^2 < \kappa(\theta) \cdot n^{-1}
\]
with
\[
\kappa(\theta) = \max\{\theta^2, E^2\} \max\left\{\frac{1}{A}, \frac{d^2}{\theta^2 (1 - C)^2}\right\}.
\]
The coefficient \(\kappa(\theta)\) always attains its least value when \(\theta^2 = E^2\); for this value of \(\theta\), one has
\[
\|E_n\|^2 < \max\left\{\frac{E^2}{A}, \frac{d^2}{(1 - C)^2}\right\} \cdot n^{-1}.
\]
Similarly, if \(E\) bounds \(\|E_0\|\), then (2.7) and (2.11) yield
\[
\|E_n\|^2 < \max\{\theta^2, E^2\} (\bar{\mu}(\theta))^n
\]
with
\[
\bar{\mu}(\theta) = 1 - \min\left\{A, \frac{(1 - C)}{(K + 1) \max\{E^2/\theta^2, 1\}}\right\}.
\]
Once again, \(\theta^2 = E^2\) yields the sharpest of these bounds, namely
\[
\|E_n\|^2 < E^2 (\bar{\mu})^n
\]
with
\[
\bar{\mu} = 1 - \min\left\{A, (1 - C) / (K + 1)^2\right\}.
\]

Note 2.2. If \(X = H\) (or a linear variety in \(H\)) and if \(T: X \to 2^X\), there is no reason for restricting \(\omega_n\) to the interval \([0, 1]\), and Theorems 2.1 and 2.2 remain valid if \(\omega\) is replaced by \(1/2(1 - C)\) in (2.4) and \(A\) is replaced by \(\frac{1}{2}\) in (2.9).

Note 2.3. Williamson has shown in [10] that
\[
\|E_0\| < E = \|Ty - y\| / 2(1 - C) \tag{2.12A}
\]

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if
\[ x_0 = y + (Ty - y)/2(1 - C) \]
(2.12B)
for some \( y \in X \). The estimate (2.12) can also be obtained by putting \( \omega = 1/2(1 - C) \) in the inequality
\[ \|x_0 - \bar{x}\|^2 < [1 - 2(1 - C)\omega]\|y - \bar{x}\|^2 + \omega^2\|Ty - y\|^2, \quad \forall \omega > 0 \]
which is gotten by the same argument that produces (2.1). For the case \( X = H \), put \( \bar{\omega} = 1/2(1 - C) \) in the rule (2.4) (see Note 2.2). Furthermore, let \( x_0 \) satisfy (2.12B) and set \( \theta^2 = E^2 \), with \( E \) given by (2.12A). Then (1.4)–(2.4) reduces to the basic iteration scheme in [10], with error bounds \( r_i \) related to the \( \beta_i \)'s in (2.3A) by
\[ r_{i+1}^2 = E^2\beta_i, \quad \forall i > 0 \]
(see Note 2.1).

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