PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 3, March 1979

A RELAXED PICARD ITERATION PROCESS FOR SET-VALUED OPERATORS OF THE MONOTONE TYPE

J. C. DUNN

ABSTRACT. The fixed points $\bar{x}$ of set-valued operators, $T: X \to 2^X$, satisfying a condition of the monotonicity type on convex subsets X of a Hilbert space are approximated by a relaxation process, $x_{n+1} = x_n + \omega_n(\tilde{T}x_n - x_n)$, in which $\tilde{T}$ is a single-valued branch of T and the relaxation parameter $\omega_n \in [0, 1]$ is made to depend in a certain way on the prior history of the process. If $\tilde{T}$ is bounded on bounded subsets of X, then $\|x_n - \bar{x}\|$ converges to 0 like $O(n^{-1/2})$. If $\tilde{T}$ is also continuous at $\bar{x}$ and if $\bar{x} = \tilde{T}\bar{x}$, then $\|x_n - \bar{x}\| = o(n^{-1/2})$. If $\tilde{T}$ satisfies a condition of the Lipschitz type at $\bar{x}$, then $\|x_n - \bar{x}\| = O(\mu^{n/2})$ for some $\mu \in [0, 1)$.

1. Introduction. Let X be a nonempty convex subset of a Hilbert space H, and suppose that for some $\bar{x} \in X$, the set-valued map $T: X \to 2^x$ satisfies the monotonicity condition,

$$1 > C_* \overset{\Delta}{=} \sup_{\substack{x \in X \\ x \neq \bar{x}}} \sup_{\xi \in Tx} \frac{\operatorname{Re}\langle \xi - \bar{x}, x - \bar{x}\rangle}{\|x - \bar{x}\|^2}; \tag{1.1}$$

(every fixed point of T necessarily coincides with $\bar{x}$). Suppose also that the range of T is bounded in the sense that

$$\operatorname{diam} \bigcup_{x \in X} Tx < \infty \tag{1.2}$$

and therefore,

$$\sup_{x \in X} \sup_{\xi \in Tx} \|\xi - \bar{x}\| < \infty. \tag{1.3}$$

If $\tilde{T}$ is any single-valued branch of T and if $\{x_n\} \subset X$ is generated by the associated relaxation process,

$$x_{n+1} = x_n + \omega_n(\tilde{T}x_n - x_n), \qquad x_0 \in X, \tag{1.4}$$

where the relaxation parameter sequence $\{\omega_n\} \subset [0, 1]$ satisfies

$$\sum_{n=0}^{\infty} \omega_n^2 < \infty \tag{1.5A}$$

Received by the editors December 27, 1977 and, in revised form, June 1, 1978.
AMS (MOS) subject classifications (1970). Primary 47H05, 47H10; Secondary 65J05, 68A10.
Key words and phrases. Set-valued Hilbert space operators, fixed points, relaxed Picard iterations, residual-dependent step lengths, continuity conditions and convergence rates.

© 1979 American Mathematical Society
0002-9939/79/0000-0106/$03.25

and

$$\sum_{n=0}^{\infty} \omega_n = \infty. \tag{1.5B}$$

Then $\{x_n\}$ converges strongly to $\bar{x}$. Moreover, for certain special parameter sequences of this type, one can show that $\|x_n - \bar{x}\| = O(n^{-1/2})$ [1], [2], [3] (for related results see [4] also). To construct such a sequence, it is enough to have a value of C satisfying $1 > C \geqslant C_*$; no additional information about T or the course of the iteration (1.4) is required, hence the parameter sequence $\{\omega_n\}$ can be computed *before* (1.4) commences.

Roughly speaking, condition (1.5) requires that $\{\omega_n\}$ converge to 0 but not "too quickly". This restriction is necessary if (1.4) is to work in the worst cases where $\tilde{T}$ is discontinuous at $\bar{x}$. Without continuity, it is possible for the residual norms $\|\tilde{T}x_n - x_n\|$ to remain bounded away from zero while x_n converges to a fixed point $\bar{x}$ of $\tilde{T}$, and this behavior is generally incompatible with (1.4) unless $\omega_n \to 0$. Unfortunately, while (1.5) insures convergence in the worst cases, it virtually guarantees a uniformly poor asymptotic rate of convergence in *all* cases, even where $\tilde{T}$ is Lipschitz continuous. On the other hand, it is known that for Lipschitz continuous $\tilde{T}$ satisfying (1.1), the sequences $\{x_n\} \subset X$ generated by (1.4) will converge *geometrically*, or linearly, to $\bar{x}$ (i.e., $\|x_n - \bar{x}\| = O(\lambda^n)$ for some $\lambda \in [0, 1)$), provided the terms in $\{\omega_n\}$ are sufficiently small but at the same time bounded away from 0 [5]–[9]. In fact, let $\tilde{T}$ satisfy the condition

$$K_* \stackrel{\Delta}{=} \sup_{\substack{x \in X \\ x \neq \bar{x}}} \frac{\|\tilde{T}x - \bar{x}\|}{\|x - \bar{x}\|} < \infty. \tag{1.6}$$

For $\omega_n \in [0, 1]$, (1.1) and (1.4) then give

$$\|x_{n+1} - \bar{x}\|^2 \leqslant \lambda_*(\omega_n)\|x_n - \bar{x}\|^2 \leqslant \lambda(\omega_n)\|x_n - \bar{x}\|^2 \tag{1.7A}$$

with

$$\lambda_*(\omega) = 1 - 2\big(1 - C_*\big)\omega + \big(1 - 2C_* + K_*^2\big)\omega^2 \tag{1.7B}$$

and

$$\lambda(\omega) = 1 - 2(1 - C)\omega + (1 - 2C + K^2)\omega^2 \tag{1.7C}$$

where

$$1 > C \geqslant C_* \tag{1.7D}$$

and

$$K \geqslant K_*. \tag{1.7E}$$

If for all $n \geqslant 0$, $\omega_n = \omega$ with ω positive and sufficiently small, the parameter $\lambda(\omega_n)$ (and a fortiori $\lambda_*(\omega_n)$) will be nonnegative and smaller than 1, hence $\{x_n\}$ will converge to $\bar{x}$ geometrically. Furthermore for any given bounds

$C \geqslant C_*$ and $K \geqslant K_*$, there is a "best" value $\omega = \omega_{\mathrm{OPT}}$ which minimizes the parameter $\lambda(\omega)$ in the bound (1.7A). To see this, observe first that for fixed K and fixed $\omega \in [0, 1]$, $\lambda(\omega)$ increases with increasing C. Furthermore, since the Schwarz inequality, (1.1) and (1.6) give $|C_*| \leqslant K_*$, it follows that $K \geqslant K_*$ is always a better estimate of C_* than any $C > K$. Therefore, consideration may be limited to C's in the range,

$$|C| \leqslant K \quad \text{and} \quad C < 1. \tag{1.7F}$$

In this event, the coefficient of ω^2 in (1.7C) is always positive and $\lambda(\omega)$ attains its least value over $\omega \in [0, 1]$ at

$$\omega_{\mathrm{OPT}} = \min\left\{1, \frac{1 - C}{1 - 2C + K^2}\right\} \tag{1.8A}$$

where

$$\lambda_{\mathrm{OPT}} = \lambda(\omega_{\mathrm{OPT}}) = \begin{cases} K^2, & \text{if } (1 - C) \geqslant 1 - 2C + K^2, \\ \dfrac{(K^2 - C^2)}{(1 - C)^2 + (K^2 - C^2)}, & \text{if } (1 - C) < 1 - 2C + K^2. \end{cases} \tag{1.8B}$$

In either case,

$$0 \leqslant \lambda_{*\,\mathrm{OPT}} \leqslant \lambda_{\mathrm{OPT}} < 1 \tag{1.8C}$$

where $\omega_{*\mathrm{OPT}}$ and $\lambda_{*\mathrm{OPT}}$ are obtained by putting $C = C^*$ and $K = K_*$ in (1.8A) and (1.8B). These estimates show that asymptotic convergence rates far better than $O(n^{-1/2})$ are attainable with (1.4) in principle when $\tilde{T}$ is sufficiently continuous at $\bar{x}$. However, in order to apply the relaxation parameter rule (1.8), one must have values for C *and* K which satisfy (1.7D) and (1.7E); if (1.8) is used with a $K < K_*$ or $C \notin [C_*, 1)$, then $\tilde{T}_{\omega_{\mathrm{OPT}}} = I + \omega_{\mathrm{OPT}}(\tilde{T} - I)$ is not necessarily contractive and the corresponding sequence $\{x_n\}$ generated by (1.4) with $\omega_n = \omega_{\mathrm{OPT}}$ may not converge to *any* limit.

The present article considers a family of iterative processes (1.4) in which ω_n is made to depend in a simply way on the prior history of x_k, for $0 \leqslant k \leqslant n$. If T is set-valued and satisfies the monotonicity condition (1.1), the corresponding error norms $\|x_n - \bar{x}\|$ are of order $O(n^{-1/2})$ as $n \to \infty$ for any branch $\tilde{T}$ that is bounded on bounded subsets of X. T need not satisfy the stronger boundedness condition (1.2) invoked in **[1]** and need not be continuous for this estimate to apply, however if $\tilde{T}$ *is* continuous at $\bar{x}$ and if $\bar{x} = \tilde{T}\bar{x}$, it turns out that $\|x_n - \bar{x}\| = o(n^{-1/2})$. Moreover, if $\tilde{T}$ satisfies condition (1.6) at $\bar{x}$, then the error norms converge to 0 *geometrically*, even though neither the constant K_* nor an upper bound K for K_* actually appears in the rule which determines ω_n (and is therefore not required for an implementation of this rule).

In **[10]**, Williamson considers a relaxation process (1.4) for $T\colon X \to 2^H$, with

ω_n depending on x_n according to a rule which turns out to be essentially a special case of the scheme proposed here; strong convergence of x_n to $\bar{x}$ is established when X is dense in H (or in a solid sphere $\subset H$), and when T is bounded on bounded subsets of X. Useful a posteriori error estimates are also derived in **[10]**, along with the basic $O(n^{-1/2})$ asymptotic convergence rate estimate.

The principal analytical tools employed in the next section were developed in **[11]** for a different purpose.

2. Results. Put

$$R_n = \tilde{T}x_n - x_n \quad \text{and} \quad E_n = x_n - \bar{x}.$$

Then (1.4) yields

$$E_{n+1} = E_n + \omega_n R_n$$

and it follows easily from the monotonicity condition (1.1) that for all $n \geqslant 0$ and $\omega_n \geqslant 0$,

$$\frac{\|E_{n+1}\|^2}{\theta^2} \leqslant \left[1 - 2(1 - C)\omega_n\right] \frac{\|E_n\|^2}{\theta^2} + \frac{\omega_n^2}{\theta^2}\|R_n\|^2 \tag{2.1}$$

where $\theta^2 \neq 0$ is a scaling parameter about which more will be said later, and where $1 > C \geqslant C_*$.

Let

$$\bar{\omega} = \min\left\{1, \frac{1}{2(1 - C)}\right\} > 0 \tag{2.2A}$$

and suppose that

$$0 \leqslant \omega_n \leqslant \bar{\omega} \leqslant 1 \tag{2.2B}$$

for all $n \geqslant 0$. Then the iterates $\{x_n\}$ generated by (1.4) cannot leave the convex set X. Moreover, the bracketed term on the right side of (2.1) is never negative and consequently a simple induction yields

$$0 \leqslant \|E_n\|^2 \leqslant B\beta_n \tag{2.3A}$$

for all $n \geqslant 0$, where,

$$B = \max\{\theta^2, \|E_0\|\} \tag{2.3B}$$

and the sequence $\{\beta_n\} \subset [0, \infty)$ is recursively generated by

$$\beta_{n+1} = \left[1 - 2(1 - C)\omega_n\right]\beta_n + \frac{\omega_n^2}{\theta^2}\|R_n\|^2, \qquad \beta_0 = 1. \tag{2.3C}$$

Since the error estimates (2.3) hold for *any* step length sequence $\{\omega_n\}$

satisfying (2.2), they are valid when ω_n is determined by the rule,

$$\omega_n = \tilde{\omega}(x_n, \beta_n; \theta) = \begin{cases} 0, & \text{if } R_n = 0, \\ \min\left\{\bar{\omega}, \dfrac{\theta^2(1-C)\beta_n}{\|R_n\|^2}\right\}, & \text{if } R_n \neq 0. \end{cases} \tag{2.4}$$

This scheme insures that $\{x_n\}$ will terminate at x_N if x_N is a fixed point of $\tilde{T}$, and otherwise minimizes the right side of (2.3C) over $\omega_n \in [0, \bar{\omega}]$, given β_n and R_n.

LEMMA 2.1. *If the sequences* $\{\omega_n\}$, $\{\beta_n\}$, *and* $\{\|R_n\|\}$ *satisfy* (2.3C) *and* (2.4) *for* $n \geqslant 0$, *then* $\{\beta_n\}$ *is nonnegative and monotone nonincreasing.*

PROOF. It has already been shown that $\beta_n \geqslant 0$ for all $n \geqslant 0$. When $\omega_n = 0$ in (2.4), (2.3C) $\Rightarrow \beta_{n+1} = \beta_n$; when $\omega_n = \theta^2(1-C)\beta_n/\|R_n\|^2$,

$$(2.3\text{C}) \Rightarrow \beta_{n+1} = \left[1 - \frac{\theta^2(1-C)^2\beta_n}{\|R_n\|^2}\right]\beta_n \leqslant \beta_n;$$

finally, when $\omega_n = \bar{\omega}$,

$$\begin{aligned}(2.3\text{C}) \Rightarrow \beta_{n+1} &= \left[1 - 2(1-C)\bar{\omega}\right]\beta_n + \bar{\omega}^2 \frac{\|R_n\|^2}{\theta^2} \\ &\leqslant \left[1 - (1-C)\bar{\omega}\right]\beta_n \leqslant \max\left\{C, \tfrac{1}{2}\right\}\beta_n < \beta_n.\end{aligned}$$

In all cases, $\beta_{n+1} \leqslant \beta_n$. Q.E.D.

COROLLARY 1. *Let* $T: X \to 2^X$ *satisfy the monotonicity condition* (1.1), *let the single-valued branch* $\tilde{T}$ *in* (1.4) *be bounded on bounded subsets of* X, *and let* $\{x_n\} \subset X$ *be generated by* (1.4)–(2.4). *Then* $\{x_n\}$ *and the associated residual sequence* $\{R_n\}$ *are bounded.*

PROOF. $\{\beta_n\}$ is monotone nonincreasing and (2.3A) and (2.3B) hold, therefore $\|x_n - \bar{x}\| \leqslant B^{1/2}$ for all $n \geqslant 0$. Since T is bounded on bounded subsets, the sequence $\{\tilde{T}x_n - \bar{x}\}$ is also bounded. Boundedness of $\{Tx_n - x_n\}$ is now immediate from the triangle inequality. Q.E.D.

LEMMA 2.2. *Let* $\{\beta_n\} \subset (0, \infty)$ *and* $\{q_n\} \subset [0, \infty)$ *satisfy*

$$\beta_{n+1} \leqslant \beta_n - q_n\beta_n^2; \qquad \beta_0 = 1 \tag{2.5}$$

for all $n \geqslant 0$. *If* $q_n \geqslant q > 0$ *for* $n \geqslant 0$, *then*

$$0 < \beta_n \leqslant \frac{1}{1+qn} = O(n^{-1}). \tag{2.6}$$

Furthermore, if $\lim_{n\to\infty} q_n = \infty$, *then* $\beta_n = o(n^{-1})$.

PROOF.[1] Put $\delta_k = 1/\beta_k$. Then for all $k \geqslant 0$, one has $\delta_k - q_k > 0$ and $\delta_{k+1} - \delta_k \geqslant q_k\delta_k/(\delta_k - q_k) \geqslant q_k$. Consequently, for all $n \geqslant 0$,

$$\delta_n = \delta_0 + \sum_{k=0}^{n-1} (\delta_{k+1} - \delta_k) \geqslant 1 + \sum_{k=0}^{n-1} q_k.$$

If $q_k \geqslant q$ for all k, one then obtains (2.6). If $q_n \to \infty$, then for any $M > 0$ there is an L so large that $k \geqslant L \Rightarrow q_k \geqslant M$. For $n > L$ one then has

$$\begin{aligned} 0 < n\beta_n &= \frac{n}{\delta_n} \\ &\leqslant \frac{n}{1 + \sum_{k=0}^{L-1} q_k + \sum_{k=L}^{n-1} q_k} \leqslant \frac{n}{(n-L)M} \end{aligned}$$

and therefore $0 \leqslant \underline{\lim}_{n\to\infty} n\beta_n \leqslant \overline{\lim}_n n\beta_n \leqslant 1/M$. Since M can be arbitrarily large, it follows that $\lim_{n\to\infty} n\beta_n = 0$. Q.E.D.

THEOREM 2.1. *Let X be a nonempty convex subset of a Hilbert space H, let T: $X \to 2^X$ satisfy* (1.1), *and let $\tilde{T}$ be a single-valued branch of T that is bounded on bounded subsets of X. Suppose that $\{x_n\} \subset X$, $\{\beta_n\} \subset [0, \infty)$, and $\{\omega_n\} \subset [0, \bar{\omega}]$ are generated by* (1.4), (2.3C), *and* (2.4) *with any $\theta \neq 0$, $C \in [C^*, 1)$, and $\bar{\omega} = \min\{1, 1/2(1 - C)\}$. Let $\{R_n\}$ and $\{E_n\}$ denote the associated residual sequence and error sequence in* (2.1). *Then $\{R_n\}$ is bounded and for all $n \geqslant 0$,*

$$0 \leqslant \|E_n\|^2 \leqslant \max\left\{\theta^2, \|E_0\|^2\right\} \cdot \beta_n \tag{2.7}$$

with $\{\beta_n\}$ nonnegative and monotone nonincreasing. Furthermore, either there is an $N \geqslant 0$ such that

$$x_n = \bar{x}, \quad \forall n \geqslant N \tag{2.8}$$

or else

$$0 < \beta_n \leqslant \frac{1}{1 + qn}, \quad \forall n \geqslant 0 \tag{2.9A}$$

with

$$q = \min\left\{A, \frac{\theta^2(1 - C)^2}{d^2}\right\} \tag{2.9B}$$

$$A = 1 - \max\left\{\tfrac{1}{2}, C\right\} \tag{2.9C}$$

and

$$\infty > d \geqslant d_* \triangleq \sup_{n > 0} \|R_n\| \tag{2.9D}$$

in which case $\|E_n\| = O(n^{-1/2})$. Finally, if $\bar{x}$ is a fixed point of $\tilde{T}$ and if $\tilde{T}$ is continuous at $\bar{x}$, then $\|x_n - \bar{x}\| = o(n^{-1/2})$.

---

[1]This result is established in [7]; for convenience, the proof is repeated here.

PROOF. Corollary 1 of Lemma 2.1 $\Rightarrow \{R_n\}$ is bounded. (2.7) has already been established in (2.3). Lemma 2.1 $\Rightarrow \{\beta_n\}$ nonnegative and monotone nonincreasing. If $R_n = 0$, then x_n is a fixed point of $\tilde{T}$ and must coincide with $\bar{x}$ because of (1.1) [**1**]. Furthermore, (1.4)–(2.4) $\Rightarrow \omega_n = 0$ and $x_{n+1} = x_n = \bar{x}$. (2.8) now follows by induction.

If $R_n \neq 0$ and $\beta_n > 0$, then (2.3C) and (2.4) give

$$0 < \beta_{n+1} \leqslant (1 - q_n\beta_n)\beta_n \tag{2.10A}$$

with

$$q_n = \min\left\{\frac{A}{\beta_n}, \frac{\theta^2(1-C)^2}{\|R_n\|^2}\right\} \geqslant q. \tag{2.10B}$$

Since $\beta = 1$, one obtains $\beta_n > 0$ for all $n \geqslant 0$ by induction if $R_n \neq 0$ for all $n \geqslant 0$. The estimate (2.9) is now immediate from Lemma 2.2.

Finally if $\bar{x} = \tilde{T}\bar{x}$, if $\tilde{T}$ is continuous at $\bar{x}$, and if $R_n \neq 0$ for all $n \geqslant 0$, then (2.7) and (2.9) $\Rightarrow \|R_n\| \to 0$ and $\beta_n \to 0$ through positive values, in which case (2.10B) $\Rightarrow \lim_{n\to\infty} q_n = \infty$. It now follows from (2.7), (2.10A) and Lemma 2.2 that $\|E_n\| = o(n^{-1/2})$. Q.E.D.

THEOREM 2.2. *Let the hypothesis of Theorem* 2.1 *hold and suppose in addition that $\tilde{T}$ satisfies* (1.6). *If $R_n \neq 0$ for $0 \leqslant n < N$, and $R_N = 0$, then* (2.8) *holds, and for $0 \leqslant n \leqslant N$,*

$$0 < \beta_n \leqslant \mu^n, \tag{2.11A}$$

with

$$\mu = 1 - \min\left\{A, \frac{\theta^2(1-C)}{B(K_* + 1)^2}\right\} \leqslant 1 - \min\left\{A, \frac{\theta^2(1-C)}{B(K+1)^2}\right\} < 1, \tag{2.11B}$$

$$A = 1 - \max\left\{\tfrac{1}{2}, C\right\}, \tag{2.11C}$$

$$B = \max\left\{\theta^2, \|E_0\|^2\right\} \tag{2.11D}$$

and

$$K \geqslant K_*. \tag{2.11E}$$

If $R_n \neq 0$ for all $n \geqslant 0$, then (2.11) *holds for all $n \geqslant 0$ and it follows from* (2.7) *that $\|E_n\| = O(\mu^{n/2})$.*

PROOF. If $R_n \neq 0$ for $0 \leqslant n < N$, then β_n is positive and satisfies (2.10) for $0 \leqslant n \leqslant N$. Furthermore,

$$\begin{aligned}(1.6) \Rightarrow \|R_n\|^2 &= \|Tx_n - x_n\|^2 \\ &\leqslant \left(\|Tx_n - \bar{x}\| + \|x_n - \bar{x}\|\right)^2 \leqslant (K+1)^2\|E_n\|^2\end{aligned}$$

and therefore (2.7) and (2.10) give

$$\frac{\theta^2(1-C)^2\beta_n}{\|R_n\|^2} \geqslant \frac{\theta^2(1-C)^2}{B(K+1)^2}$$

and

$$0 < 1 - q_n\beta_n \leqslant \mu$$

for $0 \leqslant n \leqslant N$. Q.E.D.

Note 2.1. Theorems 2.1 and 2.2 hold for any nonzero value of the scaling parameter θ. If E is any upper bound on $\|E_0\|$, then (2.7) and (2.8) give the estimate

$$\|E_n\|^2 \leqslant \kappa(\theta)\cdot n^{-1}$$

with

$$\kappa(\theta) = \max\{\theta^2, E^2\}\max\left\{\frac{1}{A}, \frac{d^2}{\theta^2(1-C)^2}\right\}.$$

The coefficient $\kappa(\theta)$ always attains its least value when $\theta^2 = E^2$; for this value of θ, one has

$$\|E_n\|^2 \leqslant \max\left\{\frac{E^2}{A}, \frac{d^2}{(1-C)^2}\right\}\cdot n^{-1}.$$

Similarly, if E bounds $\|E_0\|$, then (2.7) and (2.11) yield

$$\|E_n\|^2 \leqslant \max\{\theta^2, E^2\}(\bar{\mu}(\theta))^n$$

with

$$\bar{\mu}(\theta) = 1 - \min\left\{A, \frac{(1-C)}{(K_* + 1)^2\max\{E^2/\theta^2, 1\}}\right\}.$$

Once again, $\theta^2 = E^2$ yields the sharpest of these bounds, namely

$$\|E_n\|^2 \leqslant E^2(\bar{\mu})^n$$

with

$$\bar{\mu} = 1 - \min\left\{A, (1-C)/(K_* + 1)^2\right\}.$$

Note 2.2. If $X = H$ (or a linear variety in H) and if $T: X \to 2^X$, there is no reason for restricting ω_n to the interval [0, 1], and Theorems 2.1 and 2.2 remain valid if $\bar{\omega}$ is replaced by $1/2(1-C)$ in (2.4) and A is replaced by $\frac{1}{2}$ in (2.9).

Note 2.3. Williamson has shown in **[10]** that

$$\|E_0\| \leqslant E \stackrel{\Delta}{=} \|Ty - y\|/2(1-C) \tag{2.12A}$$

if

$$x_0 = y + (Ty - y)/2(1 - C) \tag{2.12B}$$

for some $y \in X$. The estimate (2.12) can also be obtained by putting $\omega = 1/2(1 - C)$ in the inequality

$$\|x_0 - \bar{x}\|^2 \leqslant [1 - 2(1 - C)\omega]\|y - \bar{x}\|^2 + \omega^2\|Ty - y\|^2, \quad \forall \omega \geqslant 0$$

which is gotten by the same argument that produces (2.1). For the case $X = H$, put $\bar{\omega} = 1/2(1 - C)$ in the rule (2.4) (see Note 2.2). Furthermore, let x_0 satisfy (2.12B) and set $\theta^2 = E^2$, with E given by (2.12A). Then (1.4)–(2.4) reduces to the basic iteration scheme in **[10]**, with error bounds r_i related to the β_i's in (2.3A) by

$$r_{i+1}^2 = E^2\beta_i, \quad \forall i \geqslant 0$$

(see Note 2.1).

References

1. J. C. Dunn, *Iterative construction of fixed points for multivalued operators of the monotone type*, J. Functional Analysis **27** (1978), 38–50.

2. R. E. Bruck, Jr., *The iterative solution of the equation $y \in x - Tx$ for a monotone operator T in Hilbert space*, Bull. Amer. Math. Soc. **79** (1973), 1258–1261.

3. B. E. Rhoades, *Fixed point iterations using infinite matrices*. III, in Proc. Conf. on Computing Fixed Points with Applications (Clemson Univ., June 1974), Academic Press, New York (to appear).

4. R. E. Bruck, *A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space*, J. Math. Anal. Appl. **48** (1974), 114–126.

5. E. H. Zarantonello, *Solving functional equations by contractive averaging*, U. S. Army Mathematics Research Center Technical Report #160, Wisconsin, June 1960.

6. ______, *The closure of the numerical range contains the spectrum*, Bull. Amer. Math. Soc. **70** (1964), 781–787.

7. ______, *The closure of the numerical range contains the spectrum*, Pacific J. Math. **22** (3) (1967), 575–595.

8. A. B. Bakusinskii and B. T. Poljak, *On the solution of variational inequalities*, Soviet Math. Dokl. **15** (6) (1974), 1705–1710.

9. F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

10. T. E. Williamson, Jr., *Geometric estimation of $x = Tx = 0$ for monotone-type operator T in Hilbert space*, Proc. Amer. Math. Soc. (to appear).

11. J. C. Dunn, *Rates of convergence for conditional gradient algorithms near singular and nonsingular extremals*, SIAM J. Control Optimization (to appear).

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650