SOME REMARKS ON THE DECOMPOSITION OF KERNELS

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Abstract. In a recent paper K. Lange has proved that the decomposition of a stochastic kernel into a continuous and discontinuous part yields kernels again. In the present paper, the author gives a short proof of this theorem and establishes a more general decomposition theorem. Finally, a counterexample shows that in general the Lebesgue decomposition of a kernel does not produce kernels.

K. Lange [4] applied the results of Dubins and Freedman [1] to the decomposition of kernels. Here is a short proof of Theorem 8 [4]. A counterexample shows, that in general the mentioned decompositions of kernels are no longer valid. Moreover, if the σ-field is not countably generated, Lemma 1 yields a decomposition of a kernel into a kernel which is concentrated on the atoms of a given countably generated sub-σ-field and a second kernel equal to zero on the atoms of the sub-σ-field.

Let Λ be any σ-field over a set X and Σ a countably generated sub-σ-field of Λ. The spaces of finite (positive) measures \( M(X, Λ) \) and \( M(X, Σ) \) may carry the σ-fields \( Λ^* \) and \( Σ^* \), the coarsest σ-fields so that the functions \( μ \mapsto μ(A) \) are measurable for all \( A \in Λ \) and \( Σ \) respectively (compare [1]). Every measure \( μ \in M(X, Λ) \) can uniquely be decomposed into a part \( μ_1 \) which is concentrated on a countable number of atoms of the sub-σ-field Σ, and a part \( μ_2 \), which is singular with respect to all measures with this property. For \( A \in Λ \) let \( μ_1|A \) denote the restriction of \( μ \) on \( A \) and \( μ_2|Σ \) the restriction of \( μ \) on the sub-σ-field. We can describe the value of the first part as \( μ_1(X) = (μ_1|Σ)(X) \) where on the right-hand side the index 1 denotes the atomic part of the restricted measure. Hence

\[
μ_1(Λ) = (μ_1|A)(X) = (((μ_1|Σ)|Σ))(X) \quad \text{for } A \in Λ.
\]

Lemma 1. The functions \( μ \mapsto μ_i \), \( i = 1, 2 \), from \( M(X, Λ) \) into \( M(X, Λ) \) are \( (Λ^*, Λ^*) \)-measurable.

Proof. It suffices to show that \( μ \mapsto μ_1(A) \) is \( Λ^* \)-measurable for all \( A \in Λ \). The restrictions

\[
M(X, Λ) \rightarrow M(X, Λ) \quad \text{and} \quad M(X, Λ) \rightarrow M(X, Σ)
\]

\[
μ \mapsto μ_1|A \quad \text{and} \quad μ \mapsto μ_2|Σ
\]
are \((\Lambda^*, \Lambda^*)\)-measurable and \((\Lambda^*, \Sigma^*)\)-measurable respectively. The function
\[
M(X, \Sigma) \to M(X, \Sigma)
\]
\[
\nu \mapsto \nu_1
\]
is \((\Sigma^*, \Sigma^*)\)-measurable \([1, \text{Theorem 2.12}]\) \((\nu_1\) is the atomic part of \(\nu\)). Hence
\[
\mu \mapsto \mu_{\nu} \mapsto (\mu_{\nu})_2 \mapsto ((\mu_{\nu})_2)_1 = (\mu_{\nu})_1(X) = \mu_1(A)
\]
is measurable. □

Following the arguments of \([4]\), Lemma 1 can be applied to the decomposition of kernels. Let \((Y, \Gamma)\) be any measurable space and \(y \mapsto \mu(y, \cdot)\) a kernel, i.e. a \((\Gamma, \Lambda^*)\)-measurable function from \(Y\) into \(M(X, \Lambda)\). For fixed \(y \in Y\) let
\[
\mu(y, \cdot) = \mu_1(y, \cdot) + \mu_2(y, \cdot)
\]
be the decomposition considered in Lemma 1. Plainly, \(y \mapsto \mu_i(y, \cdot), i = 1, 2,\) are kernels. We now give a short proof of a decomposition theorem of K. Lange \([4, \text{Theorem 8}]\), without using the Fell topology. For a finite measure \(\mu\) over \((\mathbb{R}^n, \mathcal{B}^n)\) let \(\mu_1\) be the part with discontinuous distribution (or \(\mu_1 = 0\)) and \(\mu_2\) the part with continuous distribution function so that \(\mu_1\) is singular with respect to all measures with continuous distribution. See \([4]\).

**Theorem 2** ([4, Theorem 8]). Let
\[
Y \to M(\mathbb{R}^n, \mathcal{B}^n)
\]
\[
y \mapsto \mu(y, \cdot)
\]
be a kernel and \(\mu(y, \cdot) = \mu_1(y, \cdot) + \mu_2(y, \cdot)\) be the decomposition described above. Then \(y \mapsto \mu_j(y, \cdot) (j = 1, 2)\) are kernels.

**Proof.** We show: \(\mu \mapsto \mu_2\) is \((\mathcal{B}^n, \mathcal{B}^n)\) measurable. Choose \(\Lambda = \mathbb{B}^n, \Sigma_i = \pi_i^{-1}(\mathbb{B})\), with \(\pi_i\) the \(i\)th projection. Let \(\mu^j_2\) be the second part of the decomposition of Lemma 1 with \(\mu^j_2 \perp \nu\) for all \(\nu\) which are concentrated on the hyperplanes given by \(\pi_i^{-1}(c))\) for \(c \in \mathbb{R}\). The function \(\varphi_j: \mu \mapsto \mu^j_2\) is measurable. Hence \(\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n\) is measurable and \(\varphi(\mu) = \mu_2\). See \([4, \text{Theorem 3}]\). □

**Remark.** The proof of Theorem 2 holds in a more general case. Let \(\Sigma\) be countably generated and for an integer \(n\) let \(\otimes_{i=1}^n \Sigma\) denote the product-σ-field on the product space \(\times_{i=1}^n X\). It may be assumed that the atoms of \(\Sigma\) consist only of single points \(x \in X\). Then the decomposition of a kernel into a part which lies on the sets \(\pi_i^{-1}(x), i = 1, \ldots, n, x \in X,\) and a second part equal to zero on those sets yields kernels.

**Example.** Let \(f\) be a measurable function from an arbitrary measurable space \((X, \Lambda)\) into a countably generated space \((Z, \Sigma')\). In the situation of Lemma 1 \(\Sigma\) can be chosen as \(f^{-1}(\Sigma')\).

For example let \(f\) from \(\mathbb{R}^n\) to \(R\) be the euclidean norm. Then for every kernel it is possible to split off that part of the kernel which gives positive mass to spheres centered at that origin.

**Counterexamples.** For countably generated measurable spaces K. Lange \([4]\) has proved a Lebesgue decomposition theorem for kernels. Here is an example which shows that this theorem is not true in general.
Choose $\mathbb{R}$ and the $\sigma$-field $\Lambda = \{ A \subset \mathbb{R} : A$ or $A^c$ countable $\}$. By definition, let $\nu$ be the measure over $(\mathbb{R}, \Lambda)$ with the property that $\nu(A) = 1$ if $A^c$ is countable and $\nu(A) = 0$ otherwise. Choose a set $E \subset \mathbb{R}$ with $E \notin \Lambda$. If we denote by $\delta_y$ the Dirac measure in the point $y$ and by $I_E$ the indicator of $E$, the function $y \mapsto \mu(y, \cdot) = I_E(y)\delta_y + I_E^c(y)\nu$ from $(\mathbb{R}, \Lambda)$ into $(M(\mathbb{R}, \Lambda), \Lambda^*)$ is a kernel, since $y \mapsto \mu(y, A)$ is $\Lambda$-measurable for all $A \in \Lambda$.

The absolutely continuous part of the Lebesgue decomposition of $\mu(y, \cdot)$ with respect to $\nu$ is $\mu_y(y, \cdot) = I_E^c(y)\nu(\cdot)$. The function $y \mapsto \mu_y(y, \cdot)$ is not a kernel, as $\mu_y(y, \mathbb{R}) = I_E^c(y)$.

The decomposition of a kernel into a discrete part and another part, whose measures are singular with respect to all measures with this property does not in general yield two kernels. Choose the kernel $\mu_y(y, \cdot)$ as above. Then the discrete part $I_E(y)\delta_y$ is not a kernel. In [3] K. Lange deals with $\Lambda^*$-measurable sets of measures. Let $L = \{ \delta_y : y \in \mathbb{R} \} \cup \{ \nu \}$ carry the restricted $\sigma$-field $\Lambda^*_L$.

The sets $\{ \delta_y : y \in \mathbb{R} \}$ and $\{ \nu \}$ are not $\Lambda^*_L$-measurable subsets of $L$. If for example $I_y : L \to \mathbb{R}$ were $\Lambda^*_L$-measurable, so would $y \mapsto I_y(\mu(y, \cdot)) = I_E^c(y)$ be measurable—a contradiction. The sets of measures $\{ \mu \in M(\mathbb{R}, \Lambda) : \mu \ll \nu \}$ and $\{ \mu \in M(\mathbb{R}, \Lambda) : \mu \perp \nu \}$ are not $\Lambda^*$-measurable, since their intersection with $L$ is not measurable with respect to the restricted $\sigma$-field. The second set is the set of discrete (finite) measures.

Other results on the decomposition of kernels will appear in [2].

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REFERENCES


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