PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 3, March 1979

THE HAAR FUNCTIONS ALMOST DIAGONALIZE MULTIPLICATION BY x

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ABSTRACT. It is shown that if A is multiplication by x on $L^2(0, 1)$, then the matrix for A given by the Haar functions has the form diagonal plus Hilbert-Schmidt.

If A is a self-adjoint operator on a complex separable Hilbert space $\mathcal{H}$, then A has the form $D + K$, where D is diagonal (i.e., the eigenvectors for D span $\mathcal{H}$) and K is a Hilbert-Schmidt operator. This fact, which was first proved by von Neumann in **[1]**, has played an important role in several areas of analysis. From the proof, however, there is no reason to expect that the basis that "almost diagonalizes" A will have a "nice" form. The purpose of this note is to show there is a natural basis (the Haar functions) for $L^2(0, 1)$ that almost diagonalizes the operator A defined by

$$(Af)(x) = xf(x), \qquad f \in L^2(0, 1).$$

If $0 \leqslant a \leqslant b \leqslant 1$, write $[a, b]$ for the characteristic function of the indicated interval. The Haar functions are defined as follows:

$$\varphi_{00} = [0, 1], \qquad \varphi_{01} = \left[0, \tfrac{1}{2}\right] - \left[\tfrac{1}{2}, 1\right]$$

and, if $n \geqslant 1, 0 \leqslant k \leqslant 2^n - 1$,

$$\varphi_{nk} = (\sqrt{2})^n\left(\left[k/2^n, k/2^n + 1/2^{n+1}\right] - \left[k/2^n + 1/2^{n+1}, (k+1)/2^n\right]\right).$$

It is obvious that the Haar functions form an orthonormal set; and, as their linear span contains the characteristic function of every dyadic subinterval, they form an orthonormal basis.

For each n and k write

$$a_{nk} = \|A\varphi_{nk}\|^2 - (A\varphi_{nk}, \varphi_{nk})^2.$$

Easy calculations show $a_{00} = a_{01} = \frac{1}{12}$ and if $n \geqslant 1, 0 \leqslant k \leqslant 2^n - 1$, then

$$\|A\varphi_{nk}\|^2 = \left(\tfrac{1}{3}\right)(2^{-2n})(3k^2 + 3k + 1) \quad \text{and} \quad (A\varphi_{nk}, \varphi_{nk}) = \left(\tfrac{1}{2}\right)(2^{-n})(2k + 1).$$

Received by the editors May 31, 1978.

AMS (MOS) subject classifications (1970). Primary 47A55, 47B15; Secondary 47B10.

Key words and phrases. Haar functions, diagonal plus Hilbert-Schmidt, multiplication by x, compact perturbation.

[1]Research supported in part by a grant from the National Science Foundation.

Therefore, $a_{nk} = (\frac{1}{12})(2^{-2n})$ and

$$a_{00} + a_{01} + \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} a_{nk} = \frac{1}{12} + \frac{1}{12} + \sum_{n=1}^{\infty} (2^n)(2^{-2n})\left(\frac{1}{12}\right) = \frac{1}{4}.$$

If K denotes the operator whose matrix in the basis $\{\varphi_{nk}\}$ is 0 on the main diagonal and agrees with the matrix for A at the off-diagonal entries, then

$$\|K\|_{HS}^2 = \sum a_{nk} = \frac{1}{4}.$$

Thus, K is a Hilbert-Schmidt operator and $D = A - K$ is diagonal.

References

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