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A SIMPLER PROOF THAT COMPACT METRIC SPACES ARE SUPERCOMPACT

CHARLES F. MILLS

ABSTRACT. We give a simpler proof that every compact metric space is supercompact.

In 1969 De Groot [5] introduced the notion of supercompactness (see definition below). All supercompact spaces are compact by the Alexander subbase lemma; the converse of this was recently shown false by Bell [1].

This leaves the question of what hypotheses on a space, beyond compactness, enable one to conclude that it is supercompact. Strok and Szymański [7] have shown, in particular, that all compact metric spaces are supercompact; their proof, however, is very complicated.

The purpose of this note is to offer a simpler proof. We have been advised that our proof is similar to one of van Douwen [4], though ours is somewhat simpler.

The author has recently shown [6] that all compact groups are supercompact.

A collection $\mathfrak{L}$ is said to be *linked* if whenever $A, B \in \mathfrak{L}$, $A \cap B \neq \varnothing$. $\mathfrak{B}$ is *binary* if for every linked $\mathfrak{L} \subset \mathfrak{B}$, $\bigcap \mathfrak{L} \neq \varnothing$. A space X is said to be *supercompact* if X has a binary closed subbase.

Fix a compact metric space X and let $\{C_n : n \in \omega\}$ be a closed base for X. We shall construct a sequence $\{\mathfrak{F}_n : n \in \omega\}$ of finite families of closed sets such that for each $n \in \omega$,

(1) $\bigcup \mathfrak{F}_m = C_n$,

(2) $\bigcup_{m \leq n} \mathfrak{F}_n$ is binary.

By (1), $\mathfrak{S} = \bigcup_{n \in \omega} \mathfrak{F}_n$ is a subbase for X; since X is compact, (2) implies that $\mathfrak{S}$ is binary.

We proceed by induction on n. Set

$$\mathfrak{F} = \bigcup_{m<n} \mathfrak{F}_m.$$

Observe that by induction hypothesis $\mathfrak{F}$ is finite and binary. For $p \in C_n$, let $\sharp p = |\{F \in \mathfrak{F} : p \in F\}|$ and let

$$A_k = \{x \in C_n : \sharp x \geqslant k\}.$$

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Note that there is k_0 such that $A_{k_0} = \varnothing$.

We define by downward induction on k a finite set $\mathcal{G}_k$ of closed subsets of X such that

(1) $A_k \subset \bigcup \mathcal{G}_k \subset C_n$;

(2) $\mathcal{G}_k \cup \mathcal{F}$ is binary;

(3) $\bigcup \mathcal{G}_k$ is a neighborhood (relative to C_n) of A_k;

(4) For every $\mathcal{Q} \subset \mathcal{F}$, either $\bigcap \mathcal{Q} \setminus \bigcup \mathcal{G}_k$ is infinite or $\bigcap \mathcal{Q} \cap (\overline{C_n \setminus \bigcup \mathcal{G}_k})$ is finite.

By (1) and (2), we may take $\mathcal{F}_n = \mathcal{G}_0$, so it remains only to construct the $\mathcal{G}_k$'s. Take $\mathcal{G}_{k_0} = \varnothing$. Fix $k < k_0$ and assume $\mathcal{G}_{k+1}$ has been constructed. Set

$$C = \overline{C_n \setminus \bigcup \mathcal{G}_{k+1}};$$

let $\mathcal{H}$ be the set of all intersections with C of intersections of precisely k distinct members of $\mathcal{F}$. By (3), $\mathcal{H}$ is a disjoint collection; also, $\mathcal{H}$ is finite. For each $\mathcal{Q} \subset \mathcal{F}$ such that $\bigcap \mathcal{Q}$ has a limit point in $C \setminus \bigcup \mathcal{H}$, pick $x_{\mathcal{Q}}$ to be such a limit. Since $\mathcal{F}$ is finite and X is normal, there is a neighborhood N of $\bigcup \mathcal{H}$ such that $\overline{N} \cap \{x_{\mathcal{Q}}: \mathcal{Q} \subset \mathcal{F} \,\&\, \bigcap \mathcal{Q}$ has a limit in $C \setminus \bigcup \mathcal{H}\} = \varnothing$. If $\mathcal{G}_k$ is chosen so that $\bigcup \mathcal{H} \subset \bigcup \mathcal{G}_k \cap C \subset \overline{N}$, (4) will be satisfied. Since $\mathcal{F} \cup \mathcal{G}_{k+1}$ is finite and X is normal, we may pick a finite disjoint cover $\mathcal{U}$ of $\bigcup \mathcal{H}$ consisting of:

(1) For each infinite $H \in \mathcal{H}$, a neighborhood U_H of H such that $\overline{U_H}$ meets no members of $\mathcal{F} \cup \mathcal{G}_{k+1}$ that do not meet H.

(2) For each finite $H \in \mathcal{H}$ and each $p \in H$, a neighborhood U_p of p such that $\overline{U_p}$ meets no members of $\mathcal{F} \cup \mathcal{G}_{k+1}$ that do not contain p.

Set $\mathcal{G}_k^0 = \{\overline{U} \cap \overline{N} \cap C: U \in \mathcal{U}\}$: if $G = \overline{U_H} \cap \overline{N} \cap C$ (respectively $G = \overline{U_p} \cap \overline{N} \cap C$) write $G = G_H$ (respectively $G = G_p$).

If $\mathcal{L} \subset \mathcal{G}_k^0 \cup \mathcal{G}_{k+1}$ and $\mathcal{L}$ is linked, then since $\mathcal{G}_k^0$ is disjoint, $\mathcal{L}$ contains at most one member G of $\mathcal{G}_k^0$. If $G = G_H$ for some $H \in \mathcal{H}$, pick $p_{\mathcal{L}} \in H \setminus \bigcup \mathcal{G}_{k+1}$ in such a way that $p_{\mathcal{M}} = p_{\mathcal{L}}$ only if $\mathcal{M} = \mathcal{L}$; this is possible since (by (4)) $H \setminus \bigcup \mathcal{G}_{k+1}$ is infinite. For $G \in \mathcal{G}_{k+1}$ let $G' = G \cup \{p_{\mathcal{L}}: G \in \mathcal{L}\}$. Observe that $G' \setminus G$ is finite, so G' is closed. Set $\mathcal{G}_k = \mathcal{G}_k^0 \cup \{G': G \in \mathcal{G}_{k+1}\}$. It is clear that (1), (3), and (4) hold of $\mathcal{G}_k$; it remains only to prove that $\mathcal{G}_k \cup \mathcal{F}$ is binary. Let $\mathcal{L} \subset \mathcal{G}_k \cup \mathcal{F}$ be linked.

Claim. $\mathcal{L}' = \{G \in \mathcal{G}_{k+1}: G' \in \mathcal{L}\}$ is linked. For assume that $G_0, G_1 \in \mathcal{L}'$. Since $G_0' \cap G_1' \neq \varnothing$ either $G_0 \cap G_1 \neq \varnothing$ or there is $p_{\mathcal{L}''} \in G_0' \cap G_1'$. But this implies that $G_0, G_1 \in \mathcal{L}''$: since $\mathcal{L}''$ is linked, $G_0 \cap G_1 \neq \varnothing$.

There are three cases to consider.

Case 1. $\mathcal{L} = \{F_0, \ldots, F_n\} \cup \{G_0', \ldots, G_m'\}$ for some $F_0, \ldots, F_n \in \mathcal{F}$.

Then $\bigcap \mathcal{L} \supset F_0 \cap \cdots \cap F_n \cap \mathcal{L}'$ which is nonempty by induction hypothesis.

Case 2. $\mathcal{L} = \{G_p\} \cup \{F_0, \ldots, F_n\} \cup \{G_0', \ldots, G_m'\}$.

Then by construction (of G_p), $p \in \bigcap \mathcal{L}$.

Case 3. $\mathcal{L} = \{G_H\} \cup \{F_0, \ldots, F_n\} \cup \{G_0', \ldots, G_m'\}$.

By choice of G_H, each F_i *contains* H; in particular, $p_{\mathcal{L}'} \in F_0 \cap \cdots \cap F_n$. Then $p_{\mathcal{L}'} \in \bigcap \mathcal{L}$. $\square$

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706