PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 3, March 1979

POINTWISE INVERSION OF THE SPHERICAL TRANSFORM ON $L^p(G/K)$, $1 \leqslant p < 2$

ROBERT J. STANTON[1] AND PETER A. TOMAS[2]

ABSTRACT. The inversion formula for the spherical Fourier transform on a noncompact symmetric space is shown to hold a.e. for $L^p(G/K)$, $1 \leqslant p < 2$.

1. Introduction. Let $X = G/K$ be a symmetric space of noncompact type. For K invariant functions on X Harish-Chandra defined the spherical transform and proved an explicit inversion formula for a class of smooth functions [4], [5] (see also [6], [2] and [8]). In [6] Helgason extended this transform to a Fourier transform on $C_c^\infty(X)$ and deduced an analogous inversion formula. The purpose of this note is to show that these inversion formulae hold a.e. for $L^p(X)$, $1 \leqslant p < 2$.

Our method of proof is classical, following the presentation in [10] for the Euclidean Fourier transform. We introduce a class of approximate identities which can be majorized by the maximal function of Clerc and Stein [1]. The pointwise inversion formula then follows as in the Euclidean case.

We begin with a brief review of some standard notation. Let G be a connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup. Let $G = KAN$ be an Iwasawa decomposition, and let $\mathfrak{A}$ be the Lie algebra of A with dimension $\mathfrak{A} = l$. Denote by P the set of roots whose restriction to $\mathfrak{A}$ is not identically zero and P^+ the positive roots in P, for some ordering. For later convenience we set $q = |P^+|/2$. Let $\mathfrak{A}_+$ ($\mathfrak{A}_+^*$) be the (dual) positive Weyl chamber and set $A_+ = \exp \mathfrak{A}_+$. Then there is the decomposition $G = K\overline{A}_+K$. Haar measures are chosen so that: dH is Lebesgue measure on $\mathfrak{A}$; $\exp^* da = dH$; dk is normalized to have mass 1; $dg = D(a)dk_1dk_2da$, where $D(\exp H) = \prod_{\alpha \in P^+}|2 \sinh \alpha(H)|$. If f is a function on A, we let $\tilde{f} = f \circ \exp$; we shall often identify functions on X with right K invariant functions on G. The negative of the Killing form gives a norm on $\mathfrak{A}$ and induces a Riemannian structure on X. For $x \in X$ (resp. $H \in \mathfrak{A}$), we let $B(x, r)$ (resp. $B(H, r)$) be the geodesic ball centered at x (resp. H) of radius r, and we let $|B(x, r)|$ denote the Riemannian measure of $B(x, r)$.

Received by the editors June 7, 1978.
AMS (MOS) subject classifications (1970). Primary 22E30, 43A30, 43A90; Secondary 33A75.
[1]Partially supported by National Science Foundation grant MCS 75-05577.
[2]Partially supported by a National Science Foundation research grant.

2. Approximate identities. In this section we develop a theory of approximate identities similar to that on Euclidean spaces. The first difficulty is defining an appropriate notion of the dilation of a function. Let ψ be a K bi-invariant function on G; then ψ is determined by its values on A and so we may define the dilated function by its values there.

DEFINITION. For $\varepsilon > 0$ and $H \in \mathfrak{A}$ set

$$\psi_\varepsilon(\exp H) = \varepsilon^{-(l+q)}\bar{D}(H)^{-1/2}\bar{D}(\varepsilon^{-1}H)^{1/2}\psi(\exp \varepsilon^{-1}H).$$

It is obvious that dilation preserves positivity and smoothness. If $\bar{\psi}$ is supported in $B(0, R)$, $\bar{\psi}_\varepsilon$ is supported in $B(0, R\varepsilon)$.

LEMMA 2.1. *If ψ is in $L^1(G)$, then $\|\psi_\varepsilon\|_1 \leqslant \|\psi\|_1$, for all $\varepsilon \leqslant 1$, and $\psi_0 = \lim_{\varepsilon\to 0}\|\psi_\varepsilon\|_1$ exists.*

PROOF.

$$\begin{aligned}\|\psi_\varepsilon\|_1 &= \int_{\mathfrak{A}} |\bar{\psi}_\varepsilon(H)|\bar{D}(H)\, dH \\ &= \int_{\mathfrak{A}} |\bar{\psi}(H)|\bar{D}^{1/2}(H)\bar{D}^{1/2}(\varepsilon H)\varepsilon^{-q}\, dH.\end{aligned}$$

Now $\bar{D}(\varepsilon H)$ is a product of factors $|\sinh \alpha(\varepsilon H)|$, and $\varepsilon^{-1}|\sinh \alpha(\varepsilon H)| \leqslant |\sinh \alpha(H)|$ whenever $\varepsilon \leqslant 1$, so that

$$\|\psi_\varepsilon\|_1 \leqslant \int |\bar{\psi}(H)|\bar{D}(H)\, dH = \|\psi\|_1.$$

Dominated convergence shows that $\lim_{\varepsilon\to 0}\|\psi_\varepsilon\|$ exists and is $\int|\bar{\psi}(H)|\bar{D}(H)^{1/2}[\prod_{\alpha\in P^+}2|\alpha(H)|]\, dH$.

In order to simplify matters somewhat we shall suppose henceforth that $\varepsilon \leqslant 1$ and that ψ is a radial function. That is, ψ is a K bi-invariant function and $\bar{\psi}$ is a radial function. We also assume ψ is smooth and $\bar{\psi}$ is supported in $B(0, 1)$; that ψ is nonnegative and that ψ_0 is 1.

Convolution on G (or on X) is given by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dy.$$

LEMMA 2.2. (a) *If f is in $C_c(X)$ then*

$$\lim_{\varepsilon\to 0} f * \psi_\varepsilon(x) = f(x), \quad \textit{for all } x \in X.$$

(b) *If f is in $L^p(X)$ for $1 \leqslant p \leqslant \infty$, then*

$$\lim_{\varepsilon\to 0}\|f * \psi_\varepsilon - f\|_p = 0.$$

PROOF. This result is standard, requiring only general properties of dilation and so we omit the proof.

DEFINITION. Let f be a locally integrable function on X; define $Mf(x) = \sup_{0<r<1}|B(x, r)|^{-1}\int_{B(x,r)}|f(y)|\, dy$.

Clerc and Stein [**1**] have shown that the map $f \mapsto Mf$ is weak (1, 1) and strong (p, p) for $p > 1$. Thus geodesic balls may be used to differentiate integrals. A point $x \in X$ is said to be in the *Lebesgue set* of f if

$$\lim_{r \to 0} |B(x, r)|^{-1} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

It follows that if f is in $L^p(X)$ for $1 \leq p \leq \infty$, the Lebesgue set of f has comeasure zero.

THEOREM 2.3. (a) *If f is locally integrable on X,*

$$\psi^* f(x) \equiv \sup_{0<r<1} |f * \psi_\varepsilon(x)| \leq c\|\psi\|_1 Mf(x).$$

(b) *If f is in $L^p(X)$ for* $1 \leq p \leq \infty$, *then* $\lim_{\varepsilon \to 0} f * \psi_\varepsilon(x) = f(x)$ *for all x in the Lebesgue set of f.*

PROOF. (a) We may assume f is positive; it suffices to prove that if $\varepsilon < 1$, $f * \psi_\varepsilon(e) \leq c\|\psi\|_1 Mf(e)$ (e the identity in G). Now ψ is a radial function which we denote by $\psi(r)$, $r > 0$; $f(x) = f(k_1 a_r k_2) = f(k_1 a_r)$ where $a_r = \exp rH$ for some $H \in \overline{\mathfrak{A}^+}$ of length 1. In fact, fixing an $H_0 \in \mathfrak{A}$ of length 1 we may express a_r as $\exp r\sigma \cdot H_0$ where σ varies over $SO(l)$. We thus write $f(x) = f(k, r, \sigma)$. Then

$$f * \psi_\varepsilon(e) = \varepsilon^{-l-q} \int_0^\infty \bar{\psi}\left(\varepsilon^{-1} r H_0\right) \int_{SO(l)} \bar{D}\left(\varepsilon^{-1} r\sigma \cdot H_0\right)^{1/2} \bar{D}(r\sigma \cdot H_0)^{1/2}$$

$$\times \int_K f(k, r, \sigma)\, dk\, d\sigma r^{l-1}\, dr.$$

A change of variables gives

$$f * \psi_\varepsilon(e) = \int_0^\infty \bar{\psi}(rH_0) \lambda(r) r^{l-1}\, dr$$

where

$$\lambda(r) = \varepsilon^{-q} \int_{SO(l)} \bar{D}(\varepsilon r\sigma \cdot H_0)^{1/2} \bar{D}(r\sigma \cdot H_0)^{1/2} \int_K f(k, r\varepsilon, \sigma) dk\, d\sigma.$$

Now $\bar{\psi}$ is a smooth function supported on $B(0, 1)$ and obviously dominated by a decreasing smooth function $\bar{\varphi}$ supported in $B(0, 1)$ with $\|\varphi\|_1 \leq c\|\psi\|_1$. Then

$$f * \psi_\varepsilon(e) \leq \int_0^1 \varphi(r) \lambda(r) r^{l-1}\, dr.$$

We set $\Lambda(r) = \int_0^r \lambda(s) s^{l-1}\, ds$ and integrate by parts getting

$$f * \psi_\varepsilon(e) \leq \varphi(r)\Lambda(r)\big]_0^1 - \int_0^1 \Lambda(r)\, d\varphi.$$

Since $\lambda(s)$ is locally integrable for each ε, $\Lambda(0) = 0$; similarly $\Lambda(1)$ is finite while $\varphi(1)$ is zero. Thus

$$f * \psi_\varepsilon(e) \leq -\int_0^1 \Lambda(r)\, d\varphi.$$

The condition $\varepsilon \leqslant 1$, we have seen, implies the inequality $\overline{D}(\varepsilon r\sigma \cdot H_0) \leqslant \varepsilon\overline{D}(r\sigma \cdot H_0)$, so that

$$\lambda(r) \leqslant \int_{SO(l)} \overline{D}(r\sigma \cdot H_0) \int_K f(k, r\varepsilon, \sigma)\, dk\, d\sigma,$$

and

$$\Lambda(r) \leqslant \varepsilon^{-l} \int_0^{r\varepsilon} \int_{SO(l)} \overline{D}(\varepsilon^{-1} s\sigma \cdot H_0) \int_K f(k, s, \sigma)\, dk\, d\sigma\, s^{l-1}\, ds.$$

Since r and ε are bounded by 1, $\overline{D}(\varepsilon^{-1}s\sigma \cdot H_0)$ is majorized by $3^{2q}\varepsilon^{-2q}\overline{D}(s\sigma \cdot H_0)$, for $s \leqslant r\varepsilon$. Therefore

$$\Lambda(r) \leqslant c\varepsilon^{-l-2q}|B(e, r\varepsilon)|Mf(e)$$

and

$$f * \psi_\varepsilon(e) \leqslant -cMf(e) \int_0^1 \varepsilon^{-l-2q}|B(e, r\varepsilon)|\, d\varphi.$$

Integrating by parts once more and noting that

$$|B(e, r\varepsilon)| = \varepsilon^l \int_0^r \int_{SO(l)} \overline{D}(\varepsilon s\sigma \cdot H_0) s^{l-1}\, d\sigma\, ds$$

we obtain

$$f * \psi_\varepsilon(e) \leqslant cMf(e) \int_0^1 \int_{SO(l)} \varepsilon^{-2q}\overline{D}(\varepsilon s\sigma \cdot H_0)\overline{\varphi}(sH_0)\, d\sigma s^{l-1}\, ds$$
$$\leqslant c\|\varphi\|_1 Mf(e) \leqslant c\|\psi\|_1 Mf(e).$$

(b) The proof of (b) uses the same integration by parts trick as in (a). Choose x in the Lebesgue set of f; given $\delta > 0$ choose r_0 such that if $r < r_0$,

$$|B(e, r)|^{-1} \int_{B(e,r)} |f(xy^{-1}) - f(x)|\, dy < \delta.$$

With φ chosen as above, choose ε_0 small enough that $\operatorname{supp} \varphi_\varepsilon \subseteq B(e, r_0)$, $\varepsilon < \varepsilon_0$. Precisely as above one shows

$$\int \varphi_\varepsilon(y)|f(xy^{-1}) - f(x)|\, dy \leqslant c\|\varphi\|_1 \delta,$$

which establishes the result.

REMARKS. (1) The class of approximate identities was chosen to make the proof of Theorem 2.3 simple, more general results will not be needed here. However, it is clear that virtually the same proof works for functions ψ that decrease sufficiently rapidly.

(2) The motivation for our definition comes from the case in which G is a complex group. For such groups the spherical functions are particularly simple, and a computation shows that the spherical transform of ψ_ε, $\hat{\psi}_\varepsilon(\lambda)$, equals $\hat{\psi}(\varepsilon\lambda)$. However, taking this as a definition of dilation for G a real group, it is clearly impossible to derive a closed formula for the dilated function ψ'_ε in terms of ψ, and so proofs involving change of variables cannot

be used. The Paley-Weiner theorem shows that supp $\psi'_\varepsilon = \varepsilon$ supp ψ, and the above program could be carried out in the presence of two additional results:

(a) if $\psi \geqslant 0$, $\psi'_\varepsilon \geqslant 0$;

(b) ψ'_ε is dominated by a sequence of radial decreasing functions with uniformly bounded L^1 norms.

Both of these seem difficult; (a), for example, seems to require an inversion formula for the map $f \mapsto F_f$. Since dilation does not preserve the property of being decreasing even for the complex groups, it is less obvious how (b) might be proved.

3. Fourier inversion. The spherical transform of a K bi-invariant function f is

$$\hat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1})\,dx,$$

where φ_λ is a zonal spherical function and λ is in $\mathfrak{A}^*_+$. For a class of smooth functions, including $C_c^\infty(K\backslash G/K)$, Harish-Chandra established the inversion formula

$$f(x) = \int_{\mathfrak{A}^*_+} \hat{f}(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}\,d\lambda.$$

The weight function $|c(\lambda)|^{-2}$ is explicitly known and of polynomial growth on $\mathfrak{A}^*$. We denote the Plancheral measure $|c(\lambda)|^{-2}\,d\lambda$ by $d\mu(\lambda)$.

Let $\hat{\psi}(\lambda)$ be in $L^1(d\mu(\lambda))$ and set

$$\psi(x) = \int_{\mathfrak{A}^*_+} \hat{\psi}(\lambda)\varphi_\lambda(x)\,d\mu(\lambda).$$

Since $|\varphi_\lambda(x)| \leqslant 1$, $\psi(x)$ is defined and in fact we have

LEMMA 3.1. *$\psi \in L^q(G)$ for all $q > 2$.*

PROOF. From the integral formula for φ_λ, $\lambda \in \mathfrak{A}^*$, we have

$$|\varphi_\lambda(x)| = \left|\int_K e^{(i\lambda-\rho)H(xk)}dk\right| \leqslant \int_K e^{-\rho H(xk)}dk \equiv \Xi(x).$$

The important inequality for $\Xi(x)$ [3, p. 279] says

$$\Xi(\exp H) \leqslant c(1+\|H\|)^d e^{-\rho(H)},$$

here c, d are positive constants. As $\bar{D}(H) = \Pi_{\alpha\in P^+}|2\sinh\alpha(H)|$ and $\rho = \frac{1}{2}\sum_{\alpha\in P^+}\alpha$, it is clear that Ξ is in $L^q(G)$ for all $q > 2$. The lemma follows from this and Holder's inequality.

PROPOSITION 3.2. *Let f be in $L^p(X)$, $1 \leqslant p < 2$, and $\hat{\psi}$ in $L^1(d\mu(\lambda))$. Then*

$$f * \psi(x) = \int_{\mathfrak{A}^*_+} \hat{\psi}(\lambda) f * \varphi_\lambda(x)\,d\mu(\lambda).$$

In particular, if f is K invariant, then

$$f * \psi(x) = \int_{\mathfrak{A}^*_+} \hat{\psi}(\lambda)\hat{f}(\lambda)\varphi_\lambda(x)\,d\mu(\lambda).$$

Proof. The first statement follows immediately from Fubini's theorem; the second is a simple computation involving the product formula for φ_λ.

Theorem 3.3. *Let f be in $L^p(X)$, $1 \leqslant p < 2$, and x be in the Lebesgue set of f. If $f * \varphi_\lambda(x)$ is in $L^1(d\mu(\lambda))$, then*

$$f(x) = \int_{\mathfrak{A}_+^*} f * \varphi_\lambda(x) d\mu(\lambda).$$

In particular, if f is K invariant and $\hat{f}$ is in $L^1(d\mu(\lambda))$, then

$$f(x) = \int_{\mathfrak{A}_+^*} \hat{f}(\lambda)\varphi_\lambda(x) d\mu(\lambda).$$

Proof. Let ψ_ε be an approximate identity as in Theorem 2.3, so that in particular ψ_ε has compact support. We observe first that the previous two results apply to ψ_ε. Indeed, the Harish transform F_{ψ_ε} is in $C_c^\infty(A)$ [6, Lemma 5.4] hence the Euclidean Fourier transform of F_{ψ_ε}, $\tilde{F}_{\psi_\varepsilon}$, is rapidly decreasing. But it is well known that $\tilde{F}_{\psi_\varepsilon} = \hat{\psi}_\varepsilon$ and so $\hat{\psi}_\varepsilon$ is in $L^1(d\mu(\lambda))$. Moreover, the inversion formula of Harish-Chandra shows that

$$\psi_\varepsilon(x) = \int_{\mathfrak{A}_+^*} \hat{\psi}_\varepsilon(\lambda)\varphi_\lambda(x) d\mu(\lambda).$$

Then if f is K invariant and in $L^p(X)$, $1 \leqslant p < 2$, Proposition 3.2 gives

$$f * \psi_\varepsilon(x) = \int_{\mathfrak{A}_+^*} \hat{f}(\lambda)\hat{\psi}_\varepsilon(\lambda)\varphi_\lambda(x) d\mu(\lambda).$$

But

$$\begin{aligned}\hat{\psi}_\varepsilon(\lambda) &= \int_{\mathfrak{A}} \psi_\varepsilon(\exp H)\varphi_\lambda(\exp - H)\overline{D}(H) dH \\ &= \int_{\mathfrak{A}} \bar{\psi}(H)\overline{D}(H)^{1/2}\overline{D}(\varepsilon H)^{1/2}\varepsilon^{-q}\varphi_\lambda(\exp - \varepsilon H) dH\end{aligned}$$

and dominated convergence shows $\lim_{\varepsilon\to 0} \hat{\psi}_\varepsilon(\lambda) = \psi_0$ (see Lemma 2.1). In addition, $|\hat{\psi}_\varepsilon(\lambda)| \leqslant \|\psi_\varepsilon\|_1 \|\varphi_\lambda\|_\infty \leqslant \|\psi\|_1$, while $\hat{f}$ is in $L^1(d\mu(\lambda))$ so that dominated convergence applies, giving

$$\lim_{\varepsilon\to 0} f * \psi_\varepsilon(x) = \int_{\mathfrak{A}_+^*} \hat{f}(\lambda)\varphi_\lambda(x)\, d\mu(\lambda).$$

But by Theorem 2.3, $\lim_{\varepsilon\to 0} f * \psi_\varepsilon(x) = f(x)$, provided x is in the Lebesgue set of f.

Next we suppose only that f is in $L^p(X)$. As before, we begin with Proposition 3.2

$$f * \psi_\varepsilon(x) = \int_{\mathfrak{A}_+^*} \hat{\psi}_\varepsilon(\lambda) f * \varphi_\lambda(x)\, d\mu(\lambda).$$

Repeating the above proof yields the result once we check that $f * |\varphi_\lambda|$ is uniformly bounded in λ. This however follows by dominating φ_λ with the basic spherical function and applying Holder's inequality.

REMARK. The inversion formula can be put into Helgason's Fourier transform notation by using [7, (14) p. 116]. In the form given in Theorem 3.3, a function is analysed into its projections in the various spherical principal series, convolution with φ_λ being the projection operator. For many questions in harmonic analysis on X this seems to be a useful definition of the Fourier transform on $L^p(X)$, $1 \leqslant p < 2$ (see [9]).

REFERENCES

1. J. L. Clerc and E. M. Stein, *L^p-multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. U. S. A. **71** (1974), 3911–3912.

2. R. Gangolli, *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. **93** (1971), 150–165.

3. Harish-Chandra, *Spherical functions on a semisimple Lie group*. I, Amer. J. Math. **80** (1958), 241–310.

4. ______, *Spherical functions on a semisimple Lie group*. II, Amer. J. Math. **80** (1958), 553–613.

5. ______, *Discrete series for semisimple Lie groups*. II, Acta Math. **116** (1966), 1–111.

6. S. Helgason, *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. **165** (1966), 297–308.

7. ______, *A duality for symmetric spaces with applications to group representations*, Advances in Math. **5** (1970), 1–154.

8. J. Rosenberg, *A quick proof of Harish-Chandra's Plancherel theorem for spherical functions on a semisimple Lie group* (preprint).

9. R. J. Stanton and P. A. Tomas, *A note on the Kunze-Stein phenomenon*, J. Functional Analysis **29** (1978), 151–159.

10. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Ser., No. 32, Princeton Univ. Press, Princeton, N. J., 1971.

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001 (Current address of R. J. Stanton)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

Current address (P. A. Tomas): Department of Mathematics, University of Texas, Austin, Texas 78712