ASYMPTOTIC STABILITY OF $\text{Ass}(M/I^nM)$

M. BRODMANN

Abstract. Let $R$ be a noetherian ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module. Let $A$ be the map from $\mathbb{N}$ into the subsets of $\text{Spec}(R)$ defined by $A(n) = \text{Ass}_R(M/I^nM)$. We shall prove:

1. For $n$ sufficiently large, $A(n)$ is independent of $n$.
   We moreover give two examples concerning the behavior of $A(n)$.

Apparently the first attempt to study the map $A$ (in the case $M = R$) was made by D. Rees [2], who proved that $P$ belongs to infinitely many sets $A(n)$, if it is prime divisor of the integral closure of a power of $I$. Ratliff [1] improved this, in showing that the above condition even implies that $P$ belongs to almost all sets $A(n)$. He moreover showed that for fixed $I$, $M = R$, there is a $k \in \mathbb{N}$, such that $P \in A(km)$ implies $P \in A(kn)$ for all large $n$, and conjectured that, in case $R$ is a domain, $k$ may be chosen to be 1 [1, PD3], e.g.

2. Conjecture. If $R = M$ is a domain, $P \in A(k)$ for some $k$ implies $P \in A(n)$ for all large $n$.

We shall see that this fails even for affine $k$-domains.

Now we prove (1). The argument occurring in [1, proof (2.11)] also applies in the case of arbitrary $M$ to show that $L = \bigcup A(n)$ is finite. But the finiteness of $L$ implies that (1) is shown, once we have proven: There is a $p \in \mathbb{N}$, such that $A(m) - A(m - 1) \subseteq A(n)$ for all $m$ and $n$ with $m < n$. In the notations

$$M_n = I^nM/I^{n+1}M, \quad B(n) = \text{Ass}_R(M_n) \quad (n > 0)$$

the short exact sequences $0 \rightarrow M_n \rightarrow M/I^{n+1}M \rightarrow M/I^nM \rightarrow 0$ show that

$$A(n + 1) - A(n) \subseteq B(n) \subseteq A(n + 1) \quad (n > 1). \quad (3)$$

So (1) is established, if we may prove that $B(n)$ is increasing for large $n$. To show this, let us first assume that $I$ contains an $M$-regular element, and accept the following result, which also occurs in [1] for $M = R$.

4. Lemma. If $((0): I)_M = (0)$, there is an $h \in \mathbb{N}$ such that

$$(I^{n+1}M: I)_M = I^nM \quad \text{for all} \quad n \geq h.$$ 

Now let $n > h - 1$ and $P \in B(n)$, and let us show that $P \in B(n + 1)$. As localization at $P$ does not affect the statement of (4), we may assume that

Received by the editors January 19, 1978 and, in revised form, June 20, 1978.


Key words and phrases. Associated prime ideals, associated graded rings and modules.

1Research on this paper was supported by "Schweizerischer Nationalfonds zur Foederung der wissenschaftlichen Forschung."

© 1979 American Mathematical Society

0002-9939/79/0000-0152/$01.75
(R, P) is local. P ∈ B(n) then implies that there is an x ∈ I^nM − I^{n+1}M such that Px ⊆ I^{n+1}M, hence PIx ⊆ I^{n+2}M. On the other hand (4) induces that x ⊆ I^{n+2}M, and we conclude that P ∈ B(n + 1).

To prove the general case we apply the above argument to I'M instead of M, which is allowed by the following reduction result.

(5) There is a t ∈ N such that ((0): I)_tM = (0).

To prove (5), put N = ((0): I)_tM. Then, by Artin-Rees, there is a t with ((0): I)_tM ⊆ IN = (0).

(6) Remark. The above reduction argument was pointed out to the author by D. Eisenbud and simplified the original proof, which used a cohomological argument to make the reduction to the case ((0): I)_tM = (0).

Next we give a proof of (4) which is more direct than the corresponding one in [1].

As Ass_R(M/(I^{n+1}M: (I)_tM)) ⊆ A(n + 1) ⊆ L, L being finite, we may restrict ourselves to prove the result for all localizations at P ∈ L, hence assume that R is local. Introduce the following associated graded objects:

\[ M = \bigoplus_{n > 0} M_n, \quad R = \bigoplus_{n > 0} R_n, \]

and let \( F_n \) be the set of leading forms in \( M \) of all elements in \( I^{n+1}M: I \) outside of \( I^nM \). We must show that \( F_n \) vanishes for all large \( n \). All the sets \( F_n \) are contained in the homogenous submodule

\[ T = \bigoplus_{n > 0} T_n = ((0): R_1)_* \]

of \( M \). Let \( t_1, \ldots, t_r \) be generating \( n \) forms of \( T \), all of degree < \( d \). Then for \( n > d \), \( T_n \subseteq R_1(t_1, \ldots, t_r)R = (0) \). This shows that \( T \) is quasi-(0) (e.g. \( T_n = (0) \) for large \( n \)), and so it suffices to show that the minimal degree \( d_n \) of all elements of \( F_n \) converges with \( n \) to infinity. So choose an \( M \)-regular element \( a \in I \). Then, by Artin-Rees, there is an \( s \in N \), such that for all \( n > s \) we have

\[ (I^{n+1}M: (I)_tM) \subseteq (I^{n+1}M: a)_M = I^{n+1}(M/a) \cap M \subseteq I^{n-1}M, \]

hence \( d_n > n - s \).

The following statement may help to decide whether a given prime \( P \) belongs to the asymptotic value \( A^* \) of \( A \).

(7) If \( I \) contains an \( M \)-regular element, then \( A \) and \( B \) coincide for large arguments.

Indeed, by (3) we only have to show that \( P \in A(n + 1) \) implies \( P \in B(n) \) for large values of \( n \). In localizing, we may assume that \( (R, P) \) is local. Then \( P \in A(n + 1) \) and (4) imply for large \( n \), \( I^{n+1}M \subseteq I^{n+1}M: P \subseteq I^nM \), hence that \( P \in B(n) \).

Using \( B(n) \subseteq A(n + 1) \) and the fact that \( B(n) \) is increasing for large \( n \) we also see that \( B \) has an asymptotic value \( B^* \).

We next give an example with \( A^* \neq B^* \).

(8) Example. Put \( R = M = K[X, Y, Z]/(XY, XZ, X^2), I = YZR, P = (X, Y, Z)_R \), where \( K \) is a field and \( X, Y, Z \) are indeterminates. Let \( J_n \) be the
preimage \((XY, XZ, X^2, Y^2Z^2)\) of \(I^n\) in \(K[X, Y, Z]\). Then \(X \not\in J_n\) and \(X(X, Y, Z) \subseteq J_n\) show that \(P \in A(n)\) for all \(n > 0\). Assume that \(P \in B(n)\) for an \(n > 0\). Then there is an \(f \in K[X, Y, Z]\) such that \(Y^nZ^n f \not\in J_{n+1}\), but \((X, Y, Z)Y^nZ^n f \subseteq J_{n+1}\). As \((X, YZ)Y^nZ^n \subseteq J_{n+1}\), we may assume that \(X\) as well as \(YZ\) do not occur in \(f\). Thus we may write \(f = u + v\), with \(u \in K[y]\) and \(v \in ZK[z]\). But now \(Y^n+1Z^n f, Y^n+1Z^n v \in J_{n+1}\) imply \(Y^n+1Z^n u \in J_{n+1}\) and \(K[y] \cap ZK[y, Z] = (Y^n+1Z^n+1, 0)\). In the same way we get that \(v = 0\). Thus, there is no \(f\) as above, which shows that \(P \not\in B(n)\) for all \(n > 0\).

Now we show that (2) is not true.

(9) Example. Let \(K\) be as above and put \(R' = K[X, Y], P = X^{m+1}R' + yR', R = K + P\). Then \(P\) is a maximal ideal of \(R\), and \(K[X^{m+1}, Y] \subseteq R \subseteq R'\) shows that \(R\) is of finite type over \(K\). \(R'/P\) obviously is a \(K = R/P\)-space of dimension \(m + 1\), which shows that \(X\) does not satisfy an integral equation of degree lower than \(m + 1\) over \(R\). As \(PR' = P\) we find \(c, d \in P\) with \(X = cR + dR, M = R\). We claim

\[
P \in A(n) \iff n < m.
\]

Obviously we have \(I^n = \sum_{i<n} c^{n-i}d^iR = c^n \sum_{i<n} X^iR\), thus \(I^n = c^nR\) for \(n > m\). This induces \(A(n) \subseteq \{Q \cap R | Q \in \text{Ass}_K(R'/c^nR')\}\). The prime divisors of \(c^nR\) are all nonmaximal, and so may not retract to \(P\). This shows the \(\Rightarrow\)-part of the above statement.

Assume now that \(P \not\in A(n)\) for \(n < m\). As none of the prime divisors of \(c^nR\) retracts to \(P\) we find an \(s \in P\) which is regular with respect to \(R/I^n\) and \(R'/c^nR\). By our choice of \(c\) and \(d\) we obviously get \(c^nR' = I^nR'\), and \(s \in P\) shows that \(R'_s = R_s\). Thus we get \(I^n = I^nR_s \cap R = I^nR'_s \cap R = c^nR' \cap R\). This induces that \(d^nX = c^nX^{n+1} \in R \cap c^nR' = I^n\). We thus find \(r_0, \ldots, r_n \in R\) with

\[
d^nX = \sum_{i=0}^n d^i c^{n-i}r_i.
\]

In dividing both sides of this equation by \(c^n\), we get

\[
X^{n+1} = \sum_{i=0}^n X^i r_i,
\]

contrary to the fact that \(X\) may not satisfy an integral equation of degree lower than \(m + 1\) over \(R\).

References


Department of Mathematics, Brandeis University, Waltham, Massachusetts 02154

Mathematisches Institut der Westfälischen Wilhelms Universität, BRD-44 Münster, West Germany (Current address)