AN ALGEBRA OF FUNCTIONS

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Abstract. It is established that the space of logarithmic potentials of \( H^1 \)-functions is closed under multiplication.

In this note we shall establish that the product of two functions, both of which are logarithmic potentials of \( H^1 \)-functions, is the logarithmic potential of a function in \( H^1 \). Here \( H^1 = H^1(R^n) \) is defined as the class of those \( L^1 \)-functions \( f \) on \( R^n \) for which it is possible to find \( L^1 \)-functions \( g_1, \ldots, g_n \) such that \( \hat{g}_j(\xi) = \frac{\xi_j}{|\xi|} \hat{f}(\xi) \), where \( \hat{f} \) denotes the Fourier transform of \( f \). The \( H^1 \)-norm of \( f \) is then

\[
\|f\|_1 = \|f\| + \sum_{i=1}^{n} \|g_i\|_1.
\]

For properties of \( H^1 \) see Stein [5] and Fefferman and Stein [3].

We define the logarithmic potential of a function \( f \) as

\[
Kf(x) = \int_{R^n} f(y) \log |x - y| \, dy.
\]

We can now state our main result.

**Theorem.** If \( f, g \in H^1 \) then there is a function \( h \in H^1 \) such that \( KfKg = Kh \). Moreover, there is a constant \( C \) independent of \( f \) and \( g \) such that \( \|h\| \leq C \|f\| \|g\| \).

We remark that in the case \( n = 1 \) this type of algebra has been studied by Coifman and Weiss [2, p. 601] in connection with the algebra of Calderon-Zygmund operators on \( R^2 \) with characteristic in \( H^1(T) \).

For other results when spaces of potentials are closed under multiplication see Strichartz [6], [7], Bagby [1].

The proof of Theorem 1 is based on the atomic decomposition of \( H^1 \)-functions. We recall that a function \( a \in L^\infty(R^n) \) is called an atom if there is a cube \( I \) with its sides parallel to the coordinate axis such that \( a \) is supported on \( I \), \( \int_R^\ast a \, dx = 0 \), and \( \|a\|_\infty \leq |I|^{-1} \). Here \( |E| \) denotes the Lebesgue measure of a set \( E \subset R^n \).

It is now known that given \( f \in H^1 \) there is a sequence of atoms \( a_j \) and
numbers $\lambda_j$ such that

$$f = \sum \lambda_j a_j$$

(1)

and $\sum |\lambda_j| < C \|f\|$ for some fixed constant $C$. Also, whenever the decomposition (1) holds $\|f\| < C \sum |\lambda_j|$ for some fixed constant $C$. For a full discussion of these results see Coifman and Weiss [2]. From these results it follows that in order to prove the theorem it is sufficient to show that if $a$ and $b$ are two atoms then $KaKb = Kh$ for some $h \in H^1$ such that $\|h\| < C$, where $C$ only depends on $n$.

We recall that a function $\phi$ is said to have bounded mean oscillation if for the family of cubes $I$ we have

$$\sup |I|^{-1} \int_I |\phi - \phi_I| \, dx = \|\phi\|_* < \infty,$$

where $\phi_I = |I|^{-1} \int_I \phi \, dx$. The space of all such functions is denoted by BMO. An important fact is that the dual of $H^1$ is BMO; see Fefferman and Stein [3]. We shall need the following estimate, whose proof is a straightforward modification of inequality (1.2) in Fefferman and Stein [3]. If $\phi \in \text{BMO}$ and $1 < p < \infty$ then for any cube $I$ centered at 0 we have

$$\left( \int \alpha (\alpha + |x|)^{-1-n} |\phi(x) - \phi_I|^p \, dx \right)^{1/p} \leq C \|\phi\|_*,$$

(2)

where $\alpha$ is the side of $I$ and $C$ is independent of $\phi$.

For a cube $I$ we denote by $2I$ the cube with the same center as $I$ but with twice the side. With the support cube of an atom $a$ we mean the smallest (axis parallel) cube which contains the support of $a$.

**Lemma.** Suppose $a$ is an atom, the support cube of which is centered at 0. There is a constant $C$ only depending on $n$ such that if $x \in 2I$ and $0 < j < n$ then

$$|\nabla_j Ka(x)| \leq C \alpha (|x| + \alpha)^{-1-j}.$$

If $x \in 2I$ and $0 < j < n - 1$ then

$$|\nabla_j Ka(x)| \leq C \alpha^{-j}.$$

Here $\alpha$ is the side of $I$ and $\nabla_j h$ denotes the partial derivatives of $h$ of order $j$ arranged in some order.

**Proof.** Let $F_j(x) = \nabla_j \log |x|$. Since $|F_j(x)| \leq C_j |x|^{-j}$, $1 < j$, it follows that if $x \not\in 2I$, then

$$\nabla_j Ka(x) = \int_I a(y) F_j(x - y) \, dy$$

$$= \int_I a(y)(F_j(x - y) - F_j(x)) \, dy.$$

Hence $|a_j Ka(x)| \leq C_j |x|^{-j-1}$ which shows the first part of the lemma. Since
\[ \| a \| < C \text{ and } \log |x| \in \text{BMO} \text{ (see John and Nirenberg [4]) it follows from the duality between } H^1 \text{ and BMO that} \]
\[ \| Ka \|_{\infty} < C. \]  

(3)

If \( 1 \leq j \leq n - 1 \) and \( x \in 2I \) then
\[ |\nabla_j Ka(x)| \leq \int |F_j(x - y)| |a(y)| dy \]
\[ \leq C \int_{|x| < \alpha} |y|^{-j/|I|^{-1}} dy \leq C\alpha^{-j}, \]

which completes the proof of the lemma.

We shall now assume that \( a \) and \( b \) are atoms with support cubes \( I \) and \( J \) respectively. We shall assume that \( a \) and \( b \) are \( C^\infty \) and we shall next show that there is a constant \( C \), only depending on \( n \), such that for all \( \phi \in \text{BMO} \) we have
\[ \left| \int_{R^n} \phi D^n (KaKb) \right| \leq C \| \phi \|_{*}. \]  

(4)

Here \( D \) denotes differentiation in the direction of a unit vector \( e \).

We let \( \alpha \) and \( \beta \) denote the sides of \( I \) and \( J \) respectively. It is no loss in generality in assuming \( \alpha < \beta \) and \( I \) is centered at the origin. Fix \( j \), \( 0 < j < n \), and put
\[ A_j = \int_{R^{n-2j}} \phi D^j KaD^{n-j} Kb. \]

Defining \( p = p_j = (n + 1)(j + 1)^{-1} \) we have
\[ 1 - p^{-1} = n(p^{-1} - kn^{-1}). \]  

(5)

From the lemma, (2) and Hölder's inequality we get
\[ |A_j| \leq C \int_{R^{n-2j}} |\alpha (|x| + \alpha)^{-j/|I|^{-1}}| \phi(x) - \phi_{2I} |D^\alpha Kb(x)| \ dx \]
\[ \leq C\alpha^{1-1/p} \| \phi \|_{*} \| D^{n-j} Kb \|_{q}, \]

where \( q = p/p - 1 \). If \( j = 0 \) then \( 1 < q < \infty \) and since
\[ \widehat{D^{n-j} Kb}(\xi) = \langle e, \xi \rangle^n \xi^{-n} b^* (\xi) \]
it follows from well-known results on multipliers (see Stein [5]) that \( \| D^n b \|_{q} \leq C \| b \|_{q} \leq C\beta^{-n(1-\alpha^{-1})} \). Here \( \langle e, \xi \rangle \) denotes the scalar product. Remembering (5) and the assumption \( \alpha < \beta \) we see that \( |A_0| \leq C \). If \( 1 < j < n - 1 \) we observe that
\[ |D^{n-j} Kb| \leq CI_j b, \]

where \( I_j f(x) = \int f(y)|x - y|^{-n} dy \). It is known (see Stein [5, Chapter V]) that if \( r \) is defined by \( q^{-1} = r^{-1} - jn^{-1} \) then \( \| I_j f \|_{q} \leq C \| f \|_{r} \). Hence \( \| D^{n-j} Kb \|_{q} \leq C\beta^{-n(1-r^{-1})} \). From (5) and the assumption \( \alpha < \beta \) it follows that \( |A_j| \leq C \).
If \( j = n \) it follows from (3) that \( |A_n| \leq C ||\phi|| \), so we have in all cases
\[
|A_j| \leq C ||\phi||, \quad 0 < j < n.
\]

We shall now put
\[
B_j = \int_{2I} (\phi - \phi_{2I}) D^jKaD^{n-j}Kb \, dx.
\]
If \( 1 < j < n - 1 \) it follows from the lemma that \( |D^jKaD^{n-j}Kb| \leq C \alpha^{-j} \beta^{j-n} \leq C |I|^{-1} \). Hence
\[
|B_j| \leq C |I|^{-1} \int_{2I} |\phi - \phi_{2I}| \, dx \leq C ||\phi||.
\]

From (3) it follows that
\[
|B_0| + |B_n| \leq C \left( \int_{2I} |\phi - \phi_{2I}|^2 \, dx \right)^{1/2} (||D^nKb|| + ||D^nKa||). 
\]
It is known that \((|I|^{-1}\int_{2I} |\phi - \phi_I|^2)^{1/2} \leq C ||\phi||_\ast\), see John and Nirenberg [4]. Since
\[
||D^nK_a|| + ||D^nK_b|| \leq C (||a|| + ||b||) 
\]
\[
\leq C (|I|^{-1/2} + |I|^{-1/2}) \leq C |I|^{-1/2}
\]

it follows that
\[
|B_j| \leq C ||\phi||, \quad 0 < j < n.
\]

Since \( \int_{R^n} D^n(\phi KaKb) \, dx = 0 \) we have
\[
\int_{R^n} \phi D^n(\phi KaKb) \, dx = \sum_{0}^{n} \binom{n}{j} (A_j + B_j),
\]
which yields (4).

To prove the theorem it is now sufficient to show that if \( F = KaKb \) then there is a function \( h \in H^1 \) such that \( \hat{h}(\xi) \leq \xi^n \tilde{F}(\xi) \) and \( ||h|| \leq C. \)

Let \( f, \ 1 < i < n, \) be defined by \( \hat{f}_i(\xi) = \xi^n \tilde{F}(\xi) \). From (4) follows \( ||f_i|| \leq C. \)

If \( T_i \) denotes the operator defined by \( T_i(f)(\xi) = \xi^n |\xi|^{-1/2} \tilde{f}(\xi) \) it follows from theory of multipliers on \( H^1 \) (see Stein [5, Chapter VIII]) that \( ||T_if|| \leq C ||f||. \)

Hence \( ||g|| \leq C, \) where \( g = \sum_i T_if_i. \) If \( T \) denotes the operator defined by \( \hat{T}(\xi) = |\xi|^{2n} \sum_i \xi_i^{2n} \tilde{f}(\xi) \), then \( T \) is bounded on \( H^1. \) Finally, observing that \( \hat{T}g(\xi) = |\xi|^{n} \tilde{F}(\xi) \) yields the theorem.

**References**


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