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A NECESSARY AND SUFFICIENT CONDITION FOR DISFOCALITY

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ABSTRACT. A criterion for the disfocality of a linear differential operator is given. The condition is analogous to a disconjugacy criterion of Pólya (1922).

1. Introduction. Consider the nth order linear differential operator L defined by

$$Ly = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y,$$

where a_i are continuous real valued functions on an open interval. The operator L is said to be *disconjugate* on an interval I if the only solution of $Ly = 0$ having n zeros or more in I counting multiplicities is the zero solution. If the only solution of $Ly = 0$ satisfying $y^{(i-1)}(t_i) = 0$ for some $t_i \in I$, $i = 1, \ldots, n$, $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, is the zero solution, then L is said to be *right disfocal* on I, and L is *left disfocal* if $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n$, $t_i \in I$. $y^{(i-1)}(t_i) = 0$ and $Ly = 0$ imply $y = 0$ on I. Pólya [4] shows that a necessary and sufficient condition for disconjugacy of L on a compact interval I is the existence of solutions $u_1, \ldots, u_{n-1}$ of $Ly = 0$ such that

$$W(u_1, \ldots, u_k) > 0, \qquad k = 1, \ldots, n-1,$$

on I, where W is the Wronskian determinant. Since it is clear from Rolle's Theorem that left or right disfocality is a stronger restriction on L than disconjugacy, it is of interest to decide if some additional simple restrictions on $u_1, \ldots, u_{n-1}$ provide a necessary and sufficient condition for left or right disfocality. A related question is considered by Nehari [3] in the context of certain generalized differential equations. In the present context one of Nehari's results (Theorem 5.1) states that, if $a_n(t) \neq 0$ for each t in an open interval I, then the only solution of $y^{(n)} + a_n(t)y = 0$ such that $y^{(i-1)}(t_i) = 0$, $t_i \in I$, $i = 1, \ldots, n$ is the zero solution if and only if all principal minors of the Wronskian matrix associated with a certain fundamental solution set are positive on I. He further shows (Theorem 7.2) that *no* minor of this matrix can vanish on I. The emphasis in the present paper is different because of the restriction on the order of the points where the successive derivatives vanish.

To motivate the results of this paper, consider an operator L of order 2. Pólya's criterion in this case is that L is disconjugate on a compact interval I

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if and only if there exists a solution u_1 of $Ly = 0$ such that $u_1 > 0$ on I; it is not difficult to see that L is right disfocal on I if and only if there exists a solution u_1 such that $u_1 > 0$, $u_1' > 0$ on I. This paper provides a generalization of the foregoing statement to operators of any order.

Throughout the paper attention is restricted to questions of right disfocality and the term "disfocal" is taken to mean "right disfocal". Analogous statements about left disfocality are obtained by formally replacing t by $-t$ throughout.

2. A disfocality criterion. In the following theorem $W(u_1, \ldots, u_n)$ denotes the Wronskian determinant $\det[u_i^{(j-1)}]$, $i, j = 1, \ldots, n$; in particular $W(u_1) = u_1$.

THEOREM. *A sufficient condition for the disfocality of L on a compact interval* $[a, b]$ *is the existence of solutions* $u_1, \ldots, u_{n-1}$ *of* $Ly = 0$ *such that*

$$W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) > 0 \tag{1}$$

for $j = 1, \ldots, n - k + 1$ *and* $k = 1, \ldots, n - 1$ *on* $[a, b)$. *A necessary condition is that solutions* $u_1, \ldots, u_n$ *exist such that* (1) *holds on* $[a, b]$ *for* $j = 1, \ldots, n - k + 1$ *and* $k = 1, \ldots, n$.

Before proving the theorem a few preliminary results will be established. Let $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n)$ denote the determinant of the $n \times n$ matrix the ith column of which is

$$\operatorname{col}\left[u_i(t_1), u_i'(t_2), \ldots, u_i^{(n-1)}(t_n)\right], \tag{2}$$

where $u_1, \ldots, u_n$ are functions for which all the derivatives involved exist. Clearly

$$\Omega(u_1, \ldots, u_n)(t, \ldots, t) = W(u_1, \ldots, u_n)(t),$$

where W is the Wronskian determinant. The symbol $\Omega(u_1, \ldots, u_n)(m; a, b)$ denotes $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n)$ when $t_1 = \cdots = t_m = a$, $t_{m+1} = \cdots = t_n = b$. Also $\Omega_j(u_1, \ldots, u_n)(t_1, t)$ denotes the determinant of the $n \times n$ matrix for which the ith column is

$$\operatorname{col}\left[u_i(t_1), u_i^{(j)}(t), u_i^{(j+1)}(t), \ldots, u_i^{(j+n-2)}(t)\right],$$

if $n > 1$, and $\Omega_j(u_1)(t_1, t) = u_1(t_1)$. Let $\beta(a) = \sup\{b\}$ such that $b > a$ and L is disfocal on $[a, b]$ and let $\alpha(b) = \inf\{a\}$ such that $a < b$ and L is disfocal on $[a, b]$. It is clear that $a < \beta(a)$ and $\alpha(b) < b$ whenever β, α exist since the existence of a nontrivial solution to

$$Ly = 0, \quad y(t_1) = y'(t_2) = \cdots = y^{(n-1)}(t_n) \tag{3}$$

is equivalent to $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0$, where $u_1, \ldots, u_n$ is a fundamental solution set; since $W(u_1, \ldots, u_n) \neq 0$ it follows by continuity that $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) \neq 0$ if $t_1, \ldots, t_n$ are sufficiently close together.

PROPOSITION 1. (a) *If $b = \beta(a)$, then L is not disfocal on $[a, b]$ and, for each nontrivial solution y of* (3) *such that $a \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant b$, the condition $t_1 = a$, $t_n = b$ holds. If a solution y exists for such a set of points $\{t_i\}$, then it is unique to within a constant multiple and t_i is the only zero in $[t_{i-1}, b]$ of $y^{(i-1)}$, $i = 1, \ldots, n$ $(t_0 = a)$.*

(b) *If $a < t_i \leqslant \cdots \leqslant t_{j-1} < b$ for one of the solutions described in* (a), *then for every choice of the $(j - i)$-tuple $(t_i, \ldots, t_{j-1})$ such that*

$$a = t_1 = \cdots = t_{i-1} \leqslant t_i \leqslant \cdots \leqslant t_{j-1} \leqslant t_j = \cdots = t_n = b$$

there exists a nontrivial solution y of (3). *In particular, there exist m, $1 \leqslant m < n$, and a nontrivial solution y to* (3) *with*

$$a = t_1 = \cdots = t_m < t_{m+1} = \cdots = t_n = b.$$

PROOF. First L is not disfocal on $[a, b]$ since, for each $c > b$, there exists a nontrivial solution y to (3) such that $a \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant c$ and hence

$$\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0$$

where $u_1, \ldots, u_n$ is a fundamental solution set. By continuity this also holds for $c = b$. If $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0$, $a \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant b$, then $t_n = b$ since $t_n < b$ contradicts the extremality of b. Also t_i is the only zero of $y^{(i-1)}$ in $[t_{i-1}, b]$ since otherwise the extremality of b is contradicted by Rolle's Theorem; similarly y is unique up to within a constant multiple since the existence of two or more linearly independent solutions implies the existence of a nontrivial solution to (3) with a zero in (t_1, b). It will be shown that if any of the points t_m $(m < n)$ is in the open interval (a, b) then the cofactors of the row $(u_1^{(m-1)}(t_m), \ldots, u_n^{(m-1)}(t_m))$ in the $n \times n$ matrix defined by (2) are all zero. From this it follows that $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0$ for all choices of t_m, in particular for all t_m satisfying $t_{m-1} \leqslant t_m \leqslant t_{m+1}$. This also shows that $t_1 = a$ since, if $t_1 \in (a, b)$, this principle implies that

$$W(u_1, \ldots, u_n)(b) = \Omega(u_1, \ldots, u_n)(b, \ldots, b) = 0$$

contradicting the fact that $u_1, \ldots, u_n$ is a fundamental solution set. Thus all the statements in Proposition 1 are proved subject to the truth of the assertion concerning the cofactors of the row $(u_1^{(m-1)}(t_m), \ldots, u_n^{(m-1)}(t_m))$. To prove this assertion, suppose the cofactors are not all zero; suppose also that $t_m < t_{m+1}$. Then the determinant of the matrix for which the ith column is

$$\operatorname{col}\left[u_i(t_1), \ldots, u_i^{(m-2)}(t_{m-1}), u_i(t), u_i^{(m)}(t_{m+1}), \ldots, u_i^{(n-1)}(t_n)\right] \tag{4}$$

is a nontrivial solution y of (3), and also $y^{(m)}(t_m) \neq 0$ since, as before, $y^{(m)}(t_m) = 0$ contradicts the extremality of b by Rolle's Theorem. Next suppose that $t_i = t_{i+1} = \cdots = t_m$ $(i \leqslant i + 1 \leqslant \cdots \leqslant m)$ and $t_j = t_{j+1} = \cdots = t_n = b$ $(m < j \leqslant j + 1 \leqslant \cdots \leqslant n)$ and that all other points (if any) in the n-tuple $(t_1, \ldots, t_n)$ are different from t_m and t_n. Let

$$Y(t, s) = \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_{i-1}, t, \ldots, t, t_{m+1}, \ldots, t_{j-1}, s, \ldots, s);$$

then $Y(t_m, t_n) = 0$ and $(\partial/\partial t)Y(t_m, t_n) = y^{(m)}(t_m) \neq 0$ so that, by the Implicit

Function Theorem, there exists a continuous function T defined on a neighbourhood of t_n such that $T(t_n) = t_m$ and $Y(T(s), s) = 0$. In particular, if s is any number in a sufficiently small neighbourhood of t_n $(= b)$, then $t_{i-1} < T(s) < t_{m+1}$ so that the extremality of b is contradicted. Thus the hypothesis that, in the matrix defined by (2), the cofactors of the row $(u_1^{(m-1)}(t_m), \ldots, u_n^{(m-1)}(t_m))$ are not all zero is false.

PROPOSITION 2. *Suppose $b = \beta(a)$ and m is minimal such that $\Omega(u_1, \ldots, u_n)(m; a, b) = 0$. Then $(\partial/\partial a)\Omega(u_1, \ldots, u_n)(m; a, b) \neq 0$ and there exists a continuously differentiable function α_m defined on a neighbourhood U of b such that $a = \alpha_m(b)$ and $\Omega(u_1, \ldots, u_n)(m; c, d) = 0$ if $c = \alpha_m(d)$; also if $d \in U$ and $d < b$ then $\alpha_m(d) < a$.*

PROOF. Under the conditions of this proposition the cofactors of the row $(u_1^{(m-1)}(a), \ldots, u_n^{(m-1)}(a))$ in the matrix for which the ith column is given by (2) with $a = t_1 = \cdots = t_m$, $t_{m+1} = \cdots = t_n = b$ are not all zero; otherwise $\Omega(u_1, \ldots, u_n)(m-1; a, b) = 0$ contradicting the minimality of m. Next let $y(t)$ be the determinant of the matrix the ith column of which is given by (4) with $a = t_1 = \cdots = t_{m-1}$, $t_{m+1} = \cdots = t_n = b$. This function is a non-trivial solution of

$$Ly = 0, \quad y(a) = \cdots = y^{(m-1)}(a) = y^{(m)}(b) = \cdots = y^{(n-1)}(b) = 0$$

which, by Proposition 1, must satisfy $y^{(m)}(a) \neq 0$. Therefore $\Omega(u_1, \ldots, u_n)(m; a, b) = 0$ and

$$(\partial/\partial a)\Omega(u_1, \ldots, u_n)(m; a, b) = y^{(m)}(a) \neq 0.$$

By the Implicit Function Theorem the existence of α_m follows. The fact that $d < b$ implies $\alpha_m(d) < a$ follows from $b = \beta(a)$.

PROPOSITION 3. *Suppose that $\beta(a)$ exists. Then $b = \beta(a)$ implies $a = \alpha(b)$. The function β is left-continuous and increasing but need not be continuous and α is continuous and nondecreasing on its domain.*

PROOF. From the definitions of α, β it follows that both functions are nondecreasing. Proposition 1 implies that β is increasing and if $b = \beta(a)$, then $a = \alpha(b)$. From Proposition 2 it follows that $\beta(a-) \geqslant \beta(a)$ and therefore, since β is increasing, $\beta(a-) = \beta(a)$. Also, since α is nondecreasing, $\beta(a-) \leqslant v \leqslant \beta(a+)$ implies $a = \alpha(v)$ and since β is increasing α is continuous. But it is not necessarily the case that $\beta(a+) = \beta(a)$. Consider the second order operator

$$Ly = W(\phi_1, \phi_2, y)/W(\phi_1, \phi_2),$$

where ϕ_1, ϕ_2 are chosen so that

$$\phi_1(2) = 0, \quad \phi_1'(t) < 0, \qquad t \in (0, 3),$$
$$\phi_2(0) = \phi_2'(1) = \phi_2'(2) = 0, \quad \phi_2'(t) > 0, \qquad t \in (0, 1) \cup (1, 2),$$
$$\phi_2'(t) < 0, \qquad t \in (2, 3).$$

Thus L is well defined since $W(\phi_1, \phi_2) \neq 0$, and $\beta(0) = 1$ because $\phi_2(0) = \phi_2'(1) = 0$ and $\phi_2'(t) > 0$, $t \in (0, 1)$; $\beta(0+) = 2$ since, if $Ly = 0$ and $y(t_0) = 0$, $t_0 \in (0, 2)$, then $y = c_1\phi_1 + c_2\phi_2$ where $c_1c_2 < 0$ so that if t_0 is sufficiently close to 0, the first zero of y' is greater than and close to 2.

PROPOSITION 4. *The operator L is disfocal on $(c, d]$ if and only if the only solution of* (3) *such that $c < t_1 < t_2 < \cdots < t_n < d$ is the zero solution.*

PROOF. The necessity of this condition is obvious. To see its sufficiency observe that if L is not disfocal on $(c, d]$, then there is a subinterval $[a, \beta(a)]$ of $(c, d]$ and, by Propositions 1 and 2, a natural number m $(1 \leqslant m < n)$ such that $\Omega(u_1, \ldots, u_n)(m; t, \beta(a))$ changes sign at $t = a$. But this cannot occur since, if $W(u_1, \ldots, u_n) > 0$, then $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) > 0$ if $c < t_1 < t_2 < \cdots < t_n < d$, and hence $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) \geqslant 0$ if $c < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant d$; in particular $\Omega(u_1, \ldots, u_n)(m; t, \beta(a)) \geqslant 0$ if $c < t \leqslant \beta(a)$.

The proof of the theorem will require the use of the following determinantal identities. If A_1, B_1, A, B, C, D are any real numbers, then

$$A\begin{vmatrix} A_1 & B_1 \\ C & D \end{vmatrix} = A_1\begin{vmatrix} A & B \\ C & D \end{vmatrix} + C\begin{vmatrix} A_1 & B_1 \\ A & B \end{vmatrix}. \tag{5}$$

Let $a_{r_1 \cdots r_m}^{s_1 \cdots s_m}$ denote the minor of the $n \times n$ matrix $[a_i^j]$ determined by the rows $r_1, \ldots, r_m$ and the columns $s_1, \ldots, s_m$ and let $b_i^j = a_{12\cdots p,p+i}^{12\cdots p,p+j}$; then

$$a_{12\cdots n}^{12\cdots n}\left(a_{12\cdots p}^{12\cdots p}\right)^{n-p-1} = b_{12\cdots n-p}^{12\cdots n-p}, \qquad p = 1, \ldots, n-1. \tag{6}$$

The relation (6) is known as Sylvester's Identity; cf. [**1**, p. 32]. It should be observed that, by row and column interchanges, any $p \times p$ minor may play the role of $a_{12\cdots p}^{12\cdots p}$.

LEMMA 1. *Suppose $u_1, \ldots, u_n$ are such that* (1) *holds on $[a, b]$ for $j = 1$ and $k = 1, \ldots, n$. Then*

$$\Omega_0(u_1, \ldots, u_k)(t_1, t) \geqslant 0, \quad k = 1, \ldots, n,$$

if $a \leqslant t_1 \leqslant t \leqslant b$, and equality holds only if $k > 1$ and $t = t_1$.

PROOF. This is a special case of Theorem V of Pólya [**4**]. Pólya outlines his proof; a detailed proof may be found in [**2**, pp. 376–378].

LEMMA 2. *The identity*

$$\begin{aligned}\Omega_j(u_1, \ldots, u_k)(t_1, t)\, W\big(u_1^{(j-1)}, \ldots, u_{k-1}^{(j-1)}\big)(t) \\ = \Omega_j(u_1, \ldots, u_{k-1})(t_1, t)\, W\big(u_1^{(j-1)}, \ldots, u_k^{(j-1)}\big)(t) \\ + \Omega_{j-1}(u_1, \ldots, u_k)(t_1, t)\, W\big(u_1^{(j)}, \ldots, u_{k-1}^{(j)}\big)(t)\end{aligned}$$

holds for all functions $u_1, \ldots, u_k$ for which these expressions exist.

PROOF. From (6) with $n = k, p = k - 2$ and

$$a_{12\cdots k}^{12\cdots k} = \Omega_j(u_1, \ldots, u_k)(t_1, t), \quad a_{12\cdots k-2}^{12\cdots k-2} = W\big(u_1^{(j)}, \ldots, u_{k-2}^{(j)}\big)(t),$$

it follows that

$$\Omega_j(u_1, \ldots, u_k)(t_1, t)\, W(u_1^{(j)}, \ldots, u_{k-2}^{(j)})(t) = \begin{vmatrix} A_1 & B_1 \\ C & D \end{vmatrix}$$

where

$$A_1 = \Omega_j(u_1, \ldots, u_{k-1})(t_1, t), \quad B_1 = \Omega_j(u_1, \ldots, u_{k-2}, u_k)(t_1, t),$$
$$C = W(u_1^{(j)}, \ldots, u_{k-1}^{(j)})(t), \quad D = W(u_1^{(j)}, \ldots, u_{k-2}^{(j)}, u_k^{(j)})(t).$$

Now Lemma 2 follows from (5) with

$$A = W(u_1^{(j-1)}, \ldots, u_{k-1}^{(j-1)})(t), \quad B = W(u_1^{(j-1)}, \ldots, u_{k-2}^{(j-1)}, u_k^{(j-1)})(t),$$

since, again by Sylvester's Identity (6),

$$W(u_1^{(j-1)}, \ldots, u_k^{(j-1)})(t)\, W(u_1^{(j)}, \ldots, u_{k-2}^{(j)})(t) = \begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

$$\Omega_{j-1}(u_1, \ldots, u_k)(t_1, t)\, W(u_1^{(j)}, \ldots, u_{k-2}^{(j)})(t) = \begin{vmatrix} A_1 & B_1 \\ A & B \end{vmatrix}.$$

PROOF OF THE THEOREM. To prove the necessity of the condition given suppose that L is disfocal on $[a, b]$. Then L is disfocal on $[\tau, b]$ for some $\tau < a$, by Proposition 3. Now let $u_1, \ldots, u_n$ be such that $Lu_i = 0$ and

$$u_i^{(j-1)}(\tau) = 0, \quad j = 1, \ldots, n - i,$$
$$(-1)^{i-1} u_i^{(n-i)}(\tau) > 0, \quad i = 1, \ldots, n. \tag{7}$$

Then $W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) \neq 0$ on $(\tau, b]$ for $j = 1, \ldots, n - k + 1$ and $k = 1, \ldots, n$ since, if $W(u_1^{(j-1)}, \ldots, u_k^{(j-1)})(\sigma) = 0$ for some $\sigma \in (\tau, b]$, there exists a nontrivial solution $y = c_1 u_1 + \cdots + c_k u_k$ of $Ly = 0$ such that

$$y(\tau) = y'(\tau) = \cdots = y^{(n-k-1)}(\tau) = 0,$$
$$y^{(j-1)}(\sigma) = y^{(j)}(\sigma) = \cdots = y^{(j+k-2)}(\sigma) = 0,$$

and by Rolle's Theorem there exist $t_1, \ldots, t_n$, $\tau \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant \sigma$, such that $y^{(j-1)}(t_j) = 0$, $j = 1, \ldots, n$, contradicting the disfocality of L on $[\tau, b]$. It remains to show that the Wronskians $W(u_1^{(j-1)}, \ldots, u_k^{(j-1)})$ are all positive on $(\tau, b]$ and hence on $[a, b]$. Condition (7) implies that, near $t = \tau$,

$$u_i^{(j-1)}(t) = [1 + o(1)]\, v_i^{(j-1)}(t), \qquad j = 1, \ldots, n - i + 1, i = 1, \ldots, n,$$

where $v_i(t) = \mu_i(t - \tau)^{n-i}$, $(-1)^{i-1}\mu_i > 0$. Therefore the Wronskians $W(u_1^{(j-1)}, \ldots, u_k^{(j-1)})$ are positive near τ and thus throughout $(\tau, b]$ for $j = 1, \ldots, n - k + 1, k = 1, \ldots, n$.

The sufficiency of the condition given in the Theorem will first be proved in the case that it holds on the closed interval $[a, b]$. Choose a solution u_n so that $W(u_1, \ldots, u_n) > 0$ on $[a, b]$, i.e. (1) holds for $k = n$ also. Under these circumstances it will be seen that

$$\Omega_j(u_1, \ldots, u_k)(t_1, t) > 0 \tag{8}$$

for $a \leqslant t_1 \leqslant t \leqslant b$, $k = 1, \ldots, n, j = 1, \ldots, n - k$. It is obvious in the case

$k = 1, j = 1, \ldots, n-1$; also, from Lemma 1, $\Omega_0(u_1, \ldots, u_k)(t_1, t) \geqslant 0$, $a \leqslant t_1 \leqslant t \leqslant b$, $k = 1, \ldots, n$. Thus, from Lemma 2, if

$$\Omega_j(u_1, \ldots, u_{k-1})(t_1, t) > 0, \quad \text{and} \quad \Omega_{j-1}(u_1, \ldots, u_k)(t_1, t) \geqslant 0,$$

then $\Omega_j(u_1, \ldots, u_k)(t_1, t) > 0$, $a \leqslant t_1 \leqslant t \leqslant b$, so that, by induction on j and k, (8) holds as stated. It will now be shown by induction on n that

$$\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) > 0, \qquad a \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant b. \tag{9}$$

This is trivial in the case $n = 1$; assume that it is true for operators of order less than n. Consider the functions $v_1, \ldots, v_{n-1}$ defined by

$$v_i(t) = \Omega_1(u_1, u_{i+1})(t_1, t), \qquad a \leqslant t_1 \leqslant t \leqslant b.$$

Since, from Sylvester's Identity (6) with $n = k + 1, p = 1$,

$$W(v_1^{(j-1)}, \ldots, v_k^{(j-1)})(t) = \Omega_j(u_1, \ldots, u_{k+1})(t_1, t)[u_1(t_1)]^{k-1}$$

it follows from (8) that

$$W(v_1^{(j-1)}, \ldots, v_k^{(j-1)}) > 0 \quad \text{on } [t_1, b], j = 1, \ldots, n-k, k = 1, \ldots, n-1,$$

and hence, by the induction hypothesis, that

$$\Omega(v_1, \ldots, v_{n-1})(t_2, \ldots, t_n) > 0, \quad \text{if } t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant b. \tag{10}$$

But, again by (6),

$$\Omega(v_1, \ldots, v_{n-1})(t_2, \ldots, t_n) = \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n)[u_1(t_1)]^{n-2}$$

and therefore from (10) the inequality (9) holds, so that the sufficiency assertion of the theorem is proved for operators of order n when (1) holds on $[a, b]$.

When (1) holds on $[a, b)$ it also holds, by continuity on $[\tau, b)$, for some $\tau < a$ and hence $\beta(\tau) \geqslant b$ from what has been already proved. Therefore $\beta(a) > b$, by Proposition 3, and L is disfocal on $[a, b]$ in this case also.

EXAMPLE. It can be seen that the operator D^n is disfocal on every interval $[a, b]$ since the functions

$$u_i(t) = (-1)^{i-1}(t - \tau)^{n-i}, \qquad i = 1, \ldots, n,$$

satisfy (1) for $j = 1, \ldots, n - k + 1$ and $k = 1, \ldots, n$ on $[a, b]$ if $\tau < a$.

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