EXTREME POINTS OF SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

H. SILVERMAN and D. N. TELAGE

Abstract. We determine coefficient bounds, distortion and covering theorems, and the extreme points for various subclasses of close-to-convex functions. All results are sharp.

1. Introduction. Let $S$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $\mathbb{U}$. A normalized function $f$ is said to be close-to-convex if there exists a function $g(z) = b_1 z + \cdots$ (Re $b_1 > 0$) (1) starlike with respect to the origin for which

$$\text{Re} \left\{ \frac{zf'}{g'} \right\} > 0 \quad (z \in \mathbb{U}).$$

(2)

It is well known [3] that the close-to-convex functions, denoted by $\mathcal{C}$, are contained in $S$.

In this paper we investigate distortion properties, coefficient bounds, and the extreme points of several subclasses of $\mathcal{C}$. A function $f$ is said to be in $\mathcal{C}_1$ if there exists a convex function $g$ of the form (1) such that (2) is satisfied. If there exists such a $g$ satisfying

$$\text{Re} \left\{ \left[ \frac{zf'}{g'} \right]' \right\} > 0 \quad (z \in \mathbb{U}),$$

then $f$ is said to be in $\mathcal{C}_2$. If

$$\text{Re} \left\{ \left[ z \frac{zf'}{g'} \right]' \left[ zg' \right]' \right\} > 0 \quad (z \in \mathbb{U}),$$

then $f$ is said to be in $\mathcal{C}_3$. In relating these classes to one another, we will rely on the following lemma due to Sakaguchi [4].

Lemma A. Let $F(z) = z + \cdots$ be analytic and $G(z) = b_1 z + \cdots$ be analytic and starlike in $\mathbb{U}$ with Re $b_1 > 0$. If Re $F'/G' > 0$ ($z \in \mathbb{U}$), then Re $F/G > 0$ ($z \in \mathbb{U}$).

Since $g$ convex implies $zg'$ is starlike, an application of Lemma A shows that $\mathcal{C}_3 \subset \mathcal{C}_2$. Reapplying Lemma A we see that $\mathcal{C}_2 \subset \mathcal{C}_1$. Further $\mathcal{C}_1 \subset \mathcal{C}$.
because convex functions are starlike. We thus have the inclusion relations
\( \mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C} \).

Geometrically, a function \( f \) is in the family \( \mathcal{C}_2 \) if \( zf' \) maps each circle \( z = re^{i\theta} \) \((r < 1)\) onto a simple closed curve whose unit tangent vector never drops back on itself more than \( \pi \) radians as \( \theta \) increases. That is, \( f \in \mathcal{C}_2 \) if and only if \( zf' \in \mathcal{C} \). The family \( \mathcal{C}_1 \), while a proper subclass of the close-to-convex functions, is not contained in the family of starlike functions. In fact, there exist functions in \( \mathcal{C}_2 \) that are not starlike. For example, the function
\[
h(z) = \frac{1 - i}{2} \frac{z}{1 - z} - \frac{1 + i}{2} \log(1 - z)
\]
is shown in the next section to be in \( \mathcal{C}_2 \). However for \( \varepsilon \) sufficiently small, \( \Re(zh'(z)/h(z)) < 0 \) when \( z = e^{i\theta}, -\varepsilon < \theta < 0 \).

2. Extreme points of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \). For a compact family \( \mathcal{F} \), we denote the closed convex hull of \( \mathcal{F} \) by \( \text{cl co} \mathcal{F} \) and the extreme points of \( \text{cl co} \mathcal{F} \) by \( \mathcal{E}(\text{cl co} \mathcal{F}) \).

**Theorem 1.** Let \( X \) be the torus \( \{(x, y) | |x| = |y| = 1\} \), \( P \) be the set of probability measures on \( X \),
\[
k(z, x, y) = (1 + x) \frac{z}{1 - yz} + xy \log(1 - yz),
\]
where \( z \in \mathbb{D} \) and \( |x| = |y| = 1 \), and let \( \mathcal{F} \) be the set of functions \( f_\mu \) defined by
\[
f_\mu(z) = \int_X k(z, x, y) \, d\mu(x, y), \quad \mu \in \mathcal{P}.
\]
Then
\[
\text{cl co} \mathcal{C}_1 = \mathcal{F}
\]
and
\[
\mathcal{E}(\text{cl co} \mathcal{C}_1) = \{ k(z, x, y) | x \neq -1 \}.
\]

**Proof.** Our proof will follow along the lines of the proof for \( \mathcal{E}(\text{cl co} \mathcal{C}) \), found in [1]. We first show that \( \text{cl co} \mathcal{C}_1 \subset \mathcal{F} \). If \( f \in \mathcal{C}_1 \), then \( p(z) = zf'(z)/g(z) \) has positive real part in \( \mathbb{D} \) for some convex function \( g \). By Herglotz' theorem, we can express \( p(z) \) as
\[
p(z) = \int_\Gamma \frac{p(0)u + \overline{p(0)}}{u - z} \, d\alpha(u)
\]
for some \( \alpha \) a probability measure on the unit circle \( \Gamma \). In [1] it is shown that we can express \( g(z) \) as
\[
g(z) = \int_\Gamma \frac{g'(0)z}{1 - vz} \, d\beta(v),
\]
where \( \beta \) is also a probability measure on \( \Gamma \). Since \( g'(0)p(0) = 1 \), we use (3), (4)
and Fubini's theorem to obtain

\[ f'(z) = \int_{\Gamma} \frac{u + g'(0)p(0)z}{u - z} \, d\alpha(u) \cdot \int_{\Gamma} \frac{1}{1 - vz} \, d\beta(v), \]

\[ = \int_{\Gamma} \frac{1 + euz}{(1 - uz)(1 - vz)} \, d\alpha(u) \, d\beta(v), \quad (5) \]

where \( e = p(0)g'(0) \) satisfies \(|e| = 1\). To show that \( f \in \mathcal{C} \) it is sufficient to show that the kernel functions in (5) belong to \( \mathcal{F}' \), the set of derivatives of functions belonging to \( \mathcal{F} \). By a theorem in [1], given \( u \) and \( v \) there is a probability measure \( \gamma \) on \( \Gamma \) such that

\[ \frac{1 + euz}{(1 - uz)(1 - vz)} = \int_{\Gamma} \frac{1 + euz}{(1 - wz)^2} \, d\gamma(w). \]

Thus we need only show for arbitrary \( w, |w| = 1 \), that we can find \( x, y, |x| = |y| = 1 \), such that

\[ \frac{d}{dz} k(z, x, y) = \frac{1 + xyz}{(1 - yz)^2} = \frac{1 + euz}{(1 - wz)^2}. \]

Choosing the unit point mass \( k(z, x, y) = k(z, e\bar{w}u, w) \), we see that \( \text{cl} \text{co} \mathcal{C}_1 \subset \mathcal{F} \).

To show that \( \mathcal{F} \subset \text{cl} \text{co} \mathcal{C}_1 \), we need only show that \( \{k(z, x, y)\} \subset \mathcal{C}_1 \) for \( |x| = |y| = 1 \). Choose a complex number \( \delta = \delta(x) \) so that \( \text{Re}\{\delta(1 + xyz)/(1 - yz)\} > 0 \). Since \( g(z) = z/\delta(1 - yz) \) is convex, we have

\[ \text{Re} \left( \frac{zdk(z, x, y)/dz}{g(z)} \right) = \text{Re} \frac{\delta(1 + xyz)}{1 - yz} > 0, \]

which shows that \( \{k(z, x, y)\} \subset \mathcal{C}_1 \).

Thus the only possible extreme points for \( \mathcal{C}_1 \) are \( \{k(z, x, y)\} \). Taking \( g = f \) in the definition of \( \mathcal{C}_1 \) and noting that convex functions are starlike, we see that \( \mathcal{C}_1 \) contains the convex functions. Since \( k(z, -1, y) = -\bar{v} \log(1 - yz) \) is convex but is not an extreme point of the closed convex hull of convex functions, it cannot be an extreme point of the larger set \( \text{cl} \text{co} \mathcal{C}_1 \).

Excluding \( x_0 = -1 \) from consideration, it suffices to show that for each \( x_0, y_0, |x_0| = |y_0| = 1 \),

\[ k(z, x_0, y_0) = \int_{X} k(z, x, y) \, d\mu(x, y) \quad (6) \]

is possible only if \( \mu \) is a unit point mass at \( (x_0, y_0) \). Differentiating both sides of (6) with respect to \( z \), we obtain

\[ \frac{1 + x_0y_0z}{(1 - y_0z)^2} = \int_{X} \frac{1 + xyz}{(1 - yz)^2} \, d\mu(x, y). \]
Setting $z = y_0 r$ and letting $r \to 1^-$, we have

$$1 + x_0 = \lim_{r \to 1^-} \int_X \left( \frac{1 - r}{1 - y y_0 r} \right)^2 (1 + x y y_0 r) \, d\mu(x, y).$$

(7)

Since the integrand in (7) is bounded by 2, we may apply the Lebesgue bounded convergence theorem to obtain

$$1 + x_0 = \int_{\Gamma \times \{y_0\}} (1 + x) \, d\mu(x, y).$$

Setting $\Gamma_0 = \Gamma \times \{y_0\}$ and $a = \mu(\Gamma_0)$, we have $0 < a < 1$ and

$$1 + x_0 = a + \int_{\Gamma_0} x \, d\mu(x, y).$$

(8)

Since $|x_0 + (1 - a)| = |\int_{\Gamma_0} x \, d\mu(x, y)| < a$ and $|x_0 + (1 - a)| \geq |x_0| - (1 - a) = a$, we must have $x_0 = -1$ or $a = 1$. Since $x_0 \neq -1$, it follows that $a = 1$. From (8) we have $x_0 = \int_{\Gamma_0} x \, d\mu(x, y)$, which can hold only if $\mu$ is a unit point mass at $(x_0, y_0)$.

**Corollary 1.** If $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in C_1$, then $|a_n| < 2 - 1/n$, with equality for $k(z, 1, -1)$.

**Proof.** We need only consider $f \in C_1$ of the form $k(z, x, y)$. It is easy to see that the modulus of the coefficients of $k$ are maximized when $x = 1$ and $y = -1$.

Similarly we have

**Corollary 2.** If $f \in C_1$, then

$$\frac{2r}{1 + r} - \log(1 + r) \leq |f(z)| \leq \frac{2r}{1 - r} + \log(1 - r) \quad (|z| < r)$$

and

$$\frac{1 - r}{(1 + r)^2} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^2} \quad (|z| < r),$$

with equality for $k(z, 1, 1)$ at $z = \pm r$.

We can use similar arguments to determine the extreme points of $C_2$. But we will use known results for $C$ to give a quicker proof.

**Theorem 2.** Let $X$ be the torus $\{(x, y) | |x| = |y| = 1\}$, $\mathcal{P}$ be the set of probability measures on $X$,

$$h(z, x, y) = \frac{1 - x y}{2} \frac{z}{1 - y z} - \frac{1 + x y}{2} y \log(1 - y z)$$

for $|x| = |y| = 1$, and let $\mathcal{F}$ be the set of functions $f_\mu$ defined by

$$f_\mu(z) = \int_X h(z, x, y) \, d\mu(x, y) \quad (\mu \in \mathcal{P}).$$
Then
\[ \text{cl co } \mathcal{C}_2 = \mathcal{F} \]
and
\[ \mathcal{O}(\text{cl co } \mathcal{C}_2) = \{h(z, x, y) | x \neq y\}. \]

**Proof.** Observe that \( f \in \mathcal{C}_2 \) if and only if \( zf' \in \mathcal{C} \). Thus the operator \( L \) defined by \( L(f) = \int_0^1 f(\xi)/\xi \, d\xi \) is a linear homeomorphism on the space of analytic functions with \( L(\mathcal{C}) = \mathcal{C}_2 \). Since
\[ h(z, x, y) = \int_0^x \frac{1 - (x + y)\xi/2}{(1 - y\xi)^2} \, d\xi, \]
the result follows from the results for \( \mathcal{C} \) proved in [1].

**Corollary.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_2 \), then \( |a_n| < 1 \) and
\[ \frac{r}{1 + r} < |f(z)| < \frac{r}{1 - r} \quad (|z| < r), \]
\[ \frac{1}{(1 + r)^2} < |f'(z)| < \frac{1}{(1 - r)^2} \quad (|z| < r). \]
Equality in all cases is obtained for \( f(z) = z/(1 - z) \).

**Remarks.** The extreme points of both \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are linear combinations of the extreme points of the convex functions and the functions convex of order \( \frac{1}{2} \). See [2]. Setting \( x = -y \), we see that the extreme points of convex functions are contained in those for \( \mathcal{C}_2 \).

### 3. The class \( \mathcal{C}_3 \)

The standard techniques cannot be applied to determine the extreme points of \( \text{cl co } \mathcal{C}_3 \) because of the presence of an additional parameter in the numerator. Nevertheless we still have sharp coefficient bounds and distortion theorems for the class \( \mathcal{C}_3 \).

**Theorem 3.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_3 \), then \( |a_n| < 2/3 + 1/3n^2 \). This result is sharp, with equality for
\[ f(z) = \frac{2}{3} \frac{z}{1 - z} - \frac{1}{3} \int_0^z \frac{\log(1 - \xi)}{\xi} \, d\xi. \]

**Proof.** If \( f \in \mathcal{C}_3 \), then there exists a convex function \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) and a function of positive real part \( p(z) = \sum_{n=0}^{\infty} c_n z^n \) with \( |b_1| = |c_0| = 1 \) such that \( [z f']' = [z g'] p \). Then
\[ [z f']' = \sum_{n=1}^{\infty} n^3 a_n z^{n-1} = \left( \sum_{n=1}^{\infty} n^2 b_n z^{n-1} \right) \left( \sum_{n=0}^{\infty} c_n z^n \right). \]
Equating coefficients, we have \( n^3 a_n = \sum_{k=1}^{n} k^2 b_k c_{n-k} \). It is well known that \( |b_n| < 1 \) and \( |c_n| < 2 \) for \( n > 1 \). Hence
\[ n^3 |a_n| < 2 \sum_{k=1}^{n-1} k^2 + n^2 = \frac{n(n - 1)(2n - 1)}{3} + n^2. \]
which simplifies to $|a_n| < \frac{2}{3} + 1/3n^2$. To show that the extremal function is in $C_3$, we take $g(z) = z/(1 - z)$.

**Theorem 4.** If $f \in C_3$, then

\[
\frac{2}{3} \frac{r}{1 + r} + \frac{1}{3} \int_0^r \log(1 + t) \frac{dt}{t} \leq |f(z)| \leq \frac{2}{3} \frac{r}{1 - r} - \frac{1}{3} \int_0^r \log(1 - t) \frac{dt}{t} \quad (|z| \leq r),
\]

\[
\frac{2}{3} \frac{1}{(1 + r)^2} + \frac{1}{3} \log(1 + r) - \frac{r}{3} \leq |f'(z)| \leq \frac{2}{3} \frac{1}{(1 - r)^2} - \frac{1}{3} \log(1 - r) \quad (0 < |z| < r).
\]

Equality holds in all cases for the extremal function of Theorem 3.

**Proof.** Setting $h = zf'$, we may write $[zh]' = pg'$, where $p(z)$ is a function of positive real part, $g(z)$ is a starlike function, and $|p(0)| = |g'(0)| = 1$. It is well known that $(1 - r)/(1 + r) < |p(z)| < (1 + r)/(1 - r)$ and $(1 - r)/(1 + r) < |g'(z)| < (1 + r)/(1 - r)$ for $|z| < r$. Hence

\[
\frac{(1 - r)^2}{(1 + r)^4} \leq |[zh'(z)]'| \leq \frac{(1 + r)^2}{(1 - r)^4} \quad (|z| \leq r).
\]

Integrating along the straight line segment from the origin to $z = re^{i\theta}$ in the right inequality of (9) we obtain

\[
|zh'(z)| \leq \int_0^r \frac{(1 + t)^2}{(1 - t)^4} \frac{dt}{3(1 - r)^3} \quad (|z| = r). \quad (10)
\]

Now for every $r$ choose $z_0$, $|z_0| = r$, such that $|h'(z_0)| = \min_{|z| = r} |h'(z)|$. If $L(z_0)$ is the pre-image of the segment $\{0, z_0h'(z_0)\}$, then

\[
|zh'(z)| > |z_0h'(z_0)| = \int_{L(z_0)} |(zh'(z))'| \, |dz| 
\geq \int_0^r \frac{(1 - t)^2}{(1 + t)^4} \frac{dt}{3(1 + r)^3}. \quad (11)
\]

In view of (10) and (11),

\[
\frac{3 + r^2}{3(1 + r)^3} \leq |[zh'(z)]'| \leq \frac{3 + r^2}{3(1 - r)^3} \quad (|z| = r).
\]

Using again the method that gave us (10) and (11), we obtain

\[
\frac{2}{3} \frac{r}{(1 + r)^2} + \frac{1}{3} \log(1 + r) \leq |zh'(z)| \leq \frac{2}{3} \frac{r}{(1 - r)^2} - \frac{1}{3} \log(1 - r).
\]
One more application yields

\[
\frac{2}{3} \frac{r}{1 + r} + \frac{1}{3} \int_0^r \frac{\log(1 + t)}{t} \, dt
\]

\[
\leq |f(z)| \leq \frac{2}{3} \frac{r}{1 - r} - \frac{1}{3} \int_0^r \frac{\log(1 - t)}{t} \, dt.
\]

The coefficient bounds give some indication as to the degree of containment of \( C_3 \subset C_2 \subset C_1 \). Another measure is the following covering theorem.

**Theorem 5.** The disk \( \mathbb{D} \) is mapped onto a domain that contains the disk \( |w| < 1 - \log 2 \approx 0.31 \) by any \( f \in C_1 \), onto a domain that contains the disk \( |w| < 0.50 \) by any \( f \in C_2 \), and onto a domain that contains the disk \( |w| < (\pi^2 + 12)/36 \approx 0.61 \) by any \( f \in C_3 \).

**Proof.** Let \( r \to 1^- \) in the lower bound of the distortion results for \( f \) in the three classes.

**References**


**Department of Mathematics, College of Charleston, Charleston, South Carolina 29401**

**Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506**