A MAXIMUM PRINCIPLE
FOR SEMILINEAR PARABOLIC SYSTEMS

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ABSTRACT. We develop a criterion insuring that every component of the solution to a system of semilinear parabolic equations is strictly positive for positive time. This criterion involves the strict (component-wise) positiveness of solutions to a related ordinary differentiable system.

In this note we present a result concerning the strict positiveness of solutions $\dot{u} = (u_1, \ldots, u_m)$ to the following system of weakly coupled parabolic equations

$$\frac{\partial}{\partial t} u_k(t, x) = L_k u_k(t, x) + F_k(t, x, \dot{u}(t, x), \Delta u_k(t, x)),$$

$$t > 0, \ x \in \Omega, \ y \in \partial \Omega, \ k = 1, \ldots, m,$$

$$B_k u_k(t, y) = 0, \ u_k(0, x) = \chi_k(x) > 0. \quad (PS)$$

Here $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, $\partial \Omega$, and $\Delta$ is the gradient operator (with respect to $x \in \mathbb{R}^n$). Also, for each $k \in \{1, \ldots, m\}$, $L_k$ is a uniformly elliptic operator with the representation

$$L_k \sim \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{k,ij}(x) \frac{\partial}{\partial x_j} \right)$$

with real-valued, smooth coefficient functions $a_{k,ij} = a_{k,ji}$; and $B_k$ is a boundary operator on $(0, \infty) \times \partial \Omega$ of the form

$$B_k u_k(t, y) = b_k(y) u_k(t, y) + \delta_k \frac{\partial}{\partial y} u_k(t, y)$$

where $\nu$ is the outward normal on $\partial \Omega$ and either $\delta_k = 0$ and $b_k(y) \equiv 1$ on $\partial \Omega$, or $\delta_k = 1$ and $b_k(y) > 0$ on $\partial \Omega$. Moreover, the real-valued function $F_k$ is $C^2$ on $[0, \infty) \times \bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^n$ and for each $R > 0$ there are numbers $M > 0$ and $\gamma \in [0, 2)$ such that

$$|F_k(t, x, \xi, \eta)| \leq M (1 + |\eta|^\gamma)$$

whenever $t \in [0, R], \ x \in \bar{\Omega}, \ \eta \in \mathbb{R}^n,$ and $\xi \in \mathbb{R}^m$ with $||\xi|| \leq R$.

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The purpose of this note is to obtain a criterion insuring the strict positiveness of every component for solutions to (PS). Our criterion is based on the behavior of solutions to a related system of ordinary differential equations in $\mathbb{R}^m$. Let $\theta$ denote the zero vector in $\mathbb{R}^m$ and for each $\xi = (\xi_i)^m_i$ and $\eta = (\eta_i)^m_i$ in $\mathbb{R}^m$ write $\xi > \eta$ only in case $\xi_i > \eta_i$ for $i = 1, \ldots, m$, and write $\xi \gg \eta$ only in case $\xi_i > \eta_i$ for $i = 1, \ldots, m$. Also, let $\theta$ denote the zero of $\mathbb{R}^m$. It is assumed that $F = (F_k)^m_k$ satisfies the following type of quasipositive condition:

$$\text{if } k \in \{1, \ldots, m\} \text{ and } \xi \gg \theta \text{ with } \xi_k = 0, \text{ then } F_k(t, x, \xi, \theta) \gg 0 \text{ for all } (t, x) \in [0, \infty) \times \Omega. \quad (1)$$

Throughout this note it is assumed that $x_0$ is a (fixed) member of $\Omega$ and that $g = (g_k)^m_k$ is defined on $[0, \infty) \times \mathbb{R}^m$ by

$$g_k(t, \xi) = F_k(t, x_0, \xi, \theta) \quad \text{for } (t, \xi) \in [0, \infty) \times \mathbb{R}^m \text{ and } k = 1, \ldots, m. \quad (2)$$

Our comparison system of ordinary differential equations is

$$z'(t) = g(t, z(t)), \quad t > 0. \quad \text{(ODE)}$$

From (1) we see that if $k \in \{1, \ldots, m\}$ and $\xi > \theta$ with $\xi_k = 0$, then $g_k(t, \xi) > \theta$, and therefore it is easily deduced that if $z$ is a solution to (ODE) with $z(t_0) > \theta$, then $z(t) > \theta$ for all $t > t_0$. Our second main supposition is

$\Gamma$ is a subset of $\{1, \ldots, m\}$ with the property that for each solution

$$z = (z_i)^m_i \text{ to (ODE), the conditions } z(t_0) > \theta \text{ and } z_i(t_0) > 0 \text{ for } i \in \Gamma \quad (3)$$

imply $z(t) \gg \theta$ for all $t > t_0$.

One should note that if $z(t_0) \gg \theta$, then $z(t) \gg \theta$ for all $t > t_0$, and hence (3) is always satisfied with $\Gamma = \{1, \ldots, m\}$. If each solution $z$ has the property that $z(t_0) > \theta$ implies $z(t) \gg \theta$ for all $t > t_0$, then (3) holds with $\Gamma$ the empty set.

The final assumption is technical, but is less restrictive than requiring $F$ to be quasimonotone:

$$\text{if } t > 0, k \in \{1, \ldots, m\} - \Gamma, \text{ and } \xi > \theta \text{ is such that } \xi_k = 0 \text{ and } F_k(t, x_0, \xi, \theta) = 0, \text{ then } F_k(t, x_0, \eta, \theta) = 0 \text{ for all } \theta < \eta < \xi. \quad (4)$$

Observe that (4) is a consequence of (1) whenever

$$\frac{\partial}{\partial \xi_j} F_i(t, x_0, \xi, \theta) > 0 \quad \text{for } i \neq j \text{ (i.e. whenever } F \text{ is quasimonotone).}$$

**THEOREM.** Suppose $\Gamma$ is a subset of $\{1, \ldots, m\}$ and (1), (3) and (4) are satisfied. Suppose also that the nonnegative initial function $\chi = (\chi_k)^m_k$ is continuous on $\overline{\Omega}$ and that $\chi_k$ is nontrivial for each $k \in \Gamma$. Then there is a $T > 0$ such that the solution $\bar{u} = (u_k)^m_k$ to (PS) exists on $[0, T) \times \Omega$ and satisfies $\bar{u}(t, x) \gg \theta$ for all $(t, x) \in (0, T) \times \Omega$.

For our proof we use the following result:

**LEMMA.** For each continuous, nonnegative $\chi = (\chi_k)^m_k$ on $\overline{\Omega}$ there is a $T > 0$ such that (PS) has a solution $\bar{u} = (u_k)^m_k$ on $[0, T) \times \Omega$ with the property that if
If \( k \in \{1, \ldots, m\} \), then either \( u_k(t, x) \equiv 0 \) on \((0, T) \times \Omega\) or \( u_k(t, x) > 0 \) on \((0, T) \times \Omega\).

We first give the proof of the theorem and then indicate the proof of this lemma.

**Proof of Theorem.** Let \( \tilde{u} = (u_k)^m \) be the solution to (PS) guaranteed by the Lemma and suppose, for contradiction, that \( u_k(t, x_i) = 0 \) for some \((t_i, x_i) \in (0, T) \times \Omega\) and some \( k \). Using the Lemma again, we see that if \( \Gamma_0 = \{ k : u_k \equiv 0 \text{ on } (0, T) \times \Omega \} \) and \( \Gamma_1 = \{ k : u_k > 0 \text{ on } (0, T) \times \Omega \} \) then \( \Gamma_0 \neq \emptyset, \Gamma_0 \cup \Gamma_1 = \{1, \ldots, m\}, \Gamma_1 \supset \Gamma \), and \( \Gamma_0 \cap \Gamma_1 = \emptyset \). It is immediate from (PS) that

\[
F_k(t, x, \tilde{u}(t, x), \theta) = 0 \quad \text{on } (0, T) \times \Omega \text{ and } k \in \Gamma_0. \tag{5}
\]

Choose \( 0 < a < b < T \) and select \( \xi = (\xi_i)^m \in \mathbb{R}^m \) such that \( \xi_i = 0 \) for \( i \in \Gamma_0 \) and \( 0 < 2\xi_i < u_i(t, x_0) \) for \( i \in \Gamma_1 \) and \( t \in [a, b] \). From (4) and (5) we have

\[
F_k(t, x_0, \eta, \theta) = 0 \quad \text{for } t \in [a, b], \theta < \eta < 2\xi_i, k \in \Gamma_0. \tag{6}
\]

Now define the function \( z = (z_i)^m \) on \([a, b]\) as follows: \( z_i(t) \equiv 0 \) on \([a, b]\) for \( i \in \Gamma_0 \) and \( \{ z_i : i \in \Gamma_1 \} \) satisfies the initial value problem

\[
z_i(t) = F_i(t, x_0, z(t), \theta), \quad t \in [a, b], z_i(a) = \xi_i, i \in \Gamma_1. \tag{7}
\]

Since \( \xi_i > 0 \) for \( i \in \Gamma_1 \), choose \( c \in (a, b) \) such that \( 0 < z_i(t) < 2\xi_i \) for all \( t \in [a, c] \) and \( i \in \Gamma_1 \), and then note that

\[
0 = F_i(t, x_0, z(t), \theta), \quad t \in [a, c], z_i(a) = \xi_i = 0, i \in \Gamma_0. \tag{7}'
\]

by (6). From the definition of \( g \) (see (2)) it is immediate from (7) and (7)' that \( z \) is a solution to (ODE) on \([a, c]\) with \( z_i(a) > 0 \) for \( i \in \Gamma_1 \supset \Gamma \) and \( z_i(t) \equiv 0 \) for \( i \in \Gamma_0 \). This contradicts assumption (3) and we conclude that \( \Gamma_0 \) must be empty. This proves the Theorem once the Lemma is established.

**Proof of Lemma.** The quasipositive assumption (1) along with a weak form of the maximum principle for parabolic equations implies that the solution \( u = (u_i)^m \) satisfies \( u(t, x) > \theta \) on \([0, T) \times \Omega\) for some \( T > 0 \) (see, e.g., the techniques in Amann [1], Lemmert [4], Lightbourne and Martin [5], and Volkmann [8]). Fix a number \( k \) in \( \{1, \ldots, m\} \) and for \( (t, x) \in [0, T) \times \Omega, r \in \mathbb{R}, \) and \( l \in \{1, \ldots, n\} \) define

\[
w_{k, l, x, r} = (\xi_i)^m \quad \text{where } \xi_k = r \text{ and } \xi_i = u_i(t, x) \text{ for } i \neq k
\]

and

\[
q_{k, l, x, r} = (\eta_i)^m \quad \text{where } \eta_l = r, \eta_j = 0 \text{ for } j < l \text{ and }
\]

\[\eta_j = \frac{\partial}{\partial x_j} u_k(t, x) \quad \text{for } j > l.\]

Now define

\[
\alpha_k(t, x) = u_k(t, x)^{-1} \int_0^{u_k(t, x)} \frac{\partial}{\partial \xi_k} F_k(t, x, w_{k, l, x, r}, \Delta u_k(t, x)) \, dr
\]
and
\[ \beta_{k,l}(t, x) = h_l(t, x)^{-1} \int_0^{h_l(t, x)} \frac{\partial}{\partial q_l} F_k(t, x, w_{k,t,x,0}, q_{k,t,x,l}) \, dq \]
where \( h_l(t, x) \equiv \frac{\partial}{\partial x_l} u_k(t, x) \) \((l = 1, \ldots, n)\). Then
\[ F_k(t, x, \bar{u}(t, x), \Delta u_k(t, x)) = F_k(t, x, w_{k,t,x,0}, \theta) + \alpha_k(t, x) u_k(t, x) + \sum_{l=1}^n \beta_{k,l}(t, x) \frac{\partial}{\partial x_l} u_k(t, x) \]
and since \( F_k(t, x, w_{k,t,x,0}, \theta) > 0 \) by (1) it follows from (PS) that
\[ \frac{\partial}{\partial t} u_k(t, x) > L_k u_k(t, x) + \alpha_k(t, x) u_k(t, x) + \sum_{l=1}^n \beta_{k,l}(t, x) \frac{\partial}{\partial x_l} u_k(t, x) \]
for all \((t, x) \in (0, T) \times \Omega\). Since \( u_k > 0 \) on \((0, T) \times \Omega\) we have from a strong form of the maximum principle \([7, pp. 173 and 175]\) that \( T \) can be chosen so that either \( u_k \equiv 0 \) on \((0, T) \times \Omega\) or \( u_k > 0 \) on \((0, T) \times \Omega\). This completes the proof indication of the Lemma.

As an illustration of this result, we consider the mathematical model of a cellular control process with either positive or negative feedback. (See Griffith \([2, 3]\).) This model is the system of three ordinary differential equations
\[
\begin{align*}
z_1' &= -\alpha z_1 + h(z_3), \quad z_1(0) > 0, \\
z_2' &= -\beta z_2 + z_1, \quad z_2(0) > 0, \\
z_3' &= -\gamma z_3 + z_2, \quad z_3(0) > 0,
\end{align*}
\]
where \( \alpha, \beta, \gamma \) are positive constants and the function \( h \) is defined on \([0, \infty)\) by either \( h(r) = r^\sigma(1 + r^\sigma)^{-1} \) (positive feedback) or \( h(r) = (1 + r^\sigma)^{-1} \) (negative feedback), and \( \sigma > 1 \) is a constant. Defining \( F = (F_i)_3 \) by the right side of (8):
\[
\begin{align*}
F_1(\xi) &= -\alpha \xi_1 + h(\xi_3); \quad F_2(\xi) = -\beta \xi_2 + \xi_1; \quad F_3(\xi) = -\gamma \xi_3 + \xi_2;
\end{align*}
\]
one may easily check that (1), (3) and (4) are satisfied whenever \( \Gamma \) is any nonempty subset of \( \{1, 2, 3\} \). From the Theorem we may conclude, for example, if at least one of the nonnegative initial values \( \chi_1, \chi_2 \) or \( \chi_3 \) is nontrivial, the solution \((u_i)_3\) to the reaction-diffusion system
\[
\begin{align*}
\frac{\partial}{\partial t} u_1 &= d_1 \Delta u_1 - \alpha u_1 + h(u_3), \\
\frac{\partial}{\partial t} u_2 &= d_2 \Delta u_2 - \beta u_2 + u_1, \quad (t, x) \in (0, \infty) \times \Omega, \\
\frac{\partial}{\partial t} u_3 &= d_3 \Delta u_3 - \gamma u_3 + u_2,
\end{align*}
\]
\((u_i)_3 = (\chi_i)_3 \) for \( t = 0, x \in \Omega\),
\[ u_i = 0 \quad \text{for} \ t > 0, y \in \partial \Omega, \text{and} \ i = 1, 2, 3, \]
satisfies \( u_i(t, x) > 0 \) for all \( t > 0, x \in \Omega \) and \( i = 1, 2, 3 \). Here \( d_i > 0 \) for \( i = 1, 2, 3 \) and \( \Delta \) is the Laplacian on \( \Omega \). Observe that the nonlinearity \( F \) is quasimonotone in the case of positive feedback, but not in the case of
negative feedback. Also, since the results of [6] depend not only on the quasimonotonicity of $F$ but also the irreducibility of the jacobian $F'(\theta)$, one sees that the results of [6] establish the strict positiveness of solutions to (9) only in the case of positive feedback with $\sigma = 1$.

**Remark.** If one assumes that (PS) has a nonnegative solution $\bar{u}$ on $[0, T) \times \Omega$, then the Theorem remains valid under less restrictive assumptions on the smoothness of $F$ and $\partial \Omega$. Note that it is necessary only to be assured that each component of $\bar{u}$ is either strictly positive or identically zero for the Theorem to hold.

**References**


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