

HIGHER ORDER ANALOGUES TO THE TANGENTIAL CAUCHY RIEMANN EQUATIONS FOR REAL SUBMANIFOLDS OF C^n WITH C.R. SINGULARITY

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ABSTRACT. An infinite succession of higher order operators are developed for real C^∞ submanifolds of C^n , with possible C.R. singularity, which reduce to the tangential Cauchy-Riemann operator in the case of a C.R. submanifold. Certain known holomorphic approximation and extension results for C.R. submanifolds are then "extended" to the non-C.R. case.

0. Introduction. In [4] the author characterized the real-analytic functions which are locally the restrictions of ambient holomorphic functions to a real-analytic submanifold of C^n with possible C.R. singularity. Theorem 4.6 of [4] gives this characterization for real-analytic submanifolds which are generic except at their C.R. singularities. The characterization is given in terms of a local parametrization for the submanifold. A main result of this discussion is Theorem 2.6 which gives the above-mentioned characterization in terms of defining functions; moreover, this approach has the advantage of admitting generalizations to C^∞ functions and C^∞ submanifolds, for example see Theorem 2.4.

§1 contains the development of the tool to be used: an infinite succession of higher order operators which reduce to the tangential Cauchy-Riemann operator in the case of a C.R. submanifold. §2 consists of theorems relating the local function theory to these operators. The application of the results from §2 is considered in §3 with several examples being provided.

1. Higher order operators. Suppose M is a C^∞ real submanifold of C^n with the origin, 0, belonging to M . Assume U is an open neighborhood of 0 and let $\rho_1, \dots, \rho_m: U \rightarrow R$ be C^∞ defining functions for M . That is, $M = \{z \in U \mid \rho_1(z) = \dots = \rho_m(z) = 0\}$ and for each $p \in M$, $d_p \rho_1 \wedge \dots \wedge d_p \rho_m \neq 0$. M is said to be a C.R. submanifold provided the complex rank of $(\bar{\partial}_p \rho_1, \dots, \bar{\partial}_p \rho_m)$ is independent of p . ($\bar{\partial}_p \rho_i$ is the complex conjugate-linear part of the Fréchet differential $d_p \rho_i$.) If the rank of $(\bar{\partial}_p \rho_1, \dots, \bar{\partial}_p \rho_m)$ is not constant for all p in any neighborhood of a point $z_0 \in M$, then z_0 is a C.R. singularity of M .

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Let $\mathcal{Q}^{(0,0)}$, respectively $\mathcal{Q}^{(0,1)}$, denote the \mathcal{C}^∞ complex valued functions on U , respectively the \mathcal{C}^∞ differential forms of type $(0, 1)$ on U . Define an operator $\bar{\partial}: \mathcal{Q}^{(0,0)} \rightarrow \mathcal{Q}^{(0,1)}$ by $f \rightarrow \bar{\partial}f$ for every $f \in \mathcal{Q}^{(0,0)}$. For any positive integer k let $\mathcal{O}^k(M)$ denote either

$$\{f \in \mathcal{Q}^{(0,0)} \mid f \text{ vanishes identically on } M \text{ to order } k\}$$

or

$$\{\beta \in \mathcal{Q}^{(0,1)} \mid \beta \text{ vanishes identically on } M \text{ to order } k\},$$

the particular meaning to be determined by context. Let $\mathcal{G}_k \subset \mathcal{Q}^{(0,1)}$ denote the span over $\mathcal{Q}^{(0,0)}$ of $\{\rho^{\alpha-j}\bar{\partial}\rho_j \mid j = 1, 2, \dots, m; |\alpha| = k \text{ and } \alpha_j > 1\} \cup \mathcal{O}^k(M)$. Here α is an m -tuple of nonnegative integers $(\alpha_1, \dots, \alpha_m)$, $|\alpha| \equiv \alpha_1 + \alpha_2 + \dots + \alpha_m$, $\rho^\alpha \equiv \rho_1^{\alpha_1} \cdot \rho_2^{\alpha_2} \cdot \dots \cdot \rho_m^{\alpha_m}$ and for any $j = 1, 2, \dots, m$, $\alpha - j \equiv (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_m)$ provided $\alpha_j \neq 0$. Define an operator $\bar{\partial}_k$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Q}^{(0,0)}/\mathcal{O}^k(M) & \xrightarrow{\bar{\partial}_k} & \mathcal{Q}^{(0,1)}/\mathcal{G}_k(M) \\ \uparrow [\cdot]_k & & \uparrow [\cdot]_k \\ \mathcal{Q}^{(0,0)} & & \mathcal{Q}^{(0,1)} \\ \cup & & \cup \\ \mathcal{O}_k(M) & \xrightarrow{\bar{\partial}} & \mathcal{G}_k(M) \end{array}$$

where $[\cdot]_k$ denotes the appropriate projection. It is a straightforward exercise to see that $\bar{\partial}_k$ is independent of the choice of defining functions. Moreover $\bar{\partial}_1$ is the induced tangential Cauchy-Riemann operator and if M is a C.R. submanifold, then $f \in \mathcal{Q}^{(0,0)}$ is a C.R. function provided $\bar{\partial}_1[f]_1 = 0$. (See Hill [5] or Greenfield [3] for this description of the induced tangential Cauchy-Riemann operator.)

2. Function theory. Suppose M , U , and $\rho_1, \dots, \rho_m: U \rightarrow R$ are as in §1. Let E denote $\{Z \in U \mid \bar{\partial}_z \rho_1 \wedge \dots \wedge \bar{\partial}_z \rho_m = 0\}$. Assume the following restrictive hypothesis on M :

Hypothesis. $E \cap M$ is nowhere dense in M .

For any $f \in \mathcal{Q}^{(0,0)}$ and positive integer k let

$${}^{k-1}[f] \equiv \{g \in [f]_1 \mid \bar{\partial}g \in \mathcal{O}^{k-1}(M)\}.$$

For each such k a calculation shows ${}^{k-1}[f] \in \mathcal{Q}^{(0,0)}/\mathcal{O}^k(M)$ (i.e. $g \in {}^{k-1}[f] \Leftrightarrow [g]_k = {}^{k-1}[f]$).

The following theorem relates the $\bar{\partial}_k$ -operators to the function theory on M .

THEOREM 2.1. *Suppose $f \in \mathcal{Q}^{(0,0)}$ and k is any positive integer. ${}^k[f] \neq \emptyset$ if and only if ${}^{k-1}[f] \neq \emptyset$ and $\bar{\partial}_k {}^{k-1}[f] = 0$.*

Given a \mathcal{C}^∞ function which has an ambient \mathcal{C}^∞ extension whose $\bar{\partial}$ vanishes on M to some prescribed order, this theorem tells when there is an extension whose $\bar{\partial}$ vanishes identically on M to one higher order. The theorem is applicable to every $f \in \mathcal{Q}^{(0,0)}$ for $k = 1$ (${}^0[f] = [f]_1$). Consequently, given k it provides a scheme for obtaining necessary and sufficient conditions for a given function $f \in \mathcal{Q}^{(0,0)}$ to have an extension $f_k \in \mathcal{Q}^{(0,0)}$ such that $\bar{\partial}f_k \in \mathcal{O}^k(M)$.

PROOF OF THEOREM 2.1. “Only if”: The proof is trivial. “If”: Suppose $g \in {}^{k-1}[f]$ and $\bar{\partial}_k {}^{k-1}[g] = 0$. Let

$$\bar{\partial}g = \sum_{j=1}^m \sum_{\substack{|\alpha|=k \\ \alpha_j \neq 0}} b_\alpha^j \rho^{\alpha-j} \bar{\partial} \rho_j + \mathcal{O}^k(M). \tag{2.1.1}$$

Thus

$$0 = \bar{\partial} \bar{\partial}g = \sum_{j,l=1}^m \sum_{\substack{|\alpha|=k \\ \alpha_j, \alpha_l \neq 0}} \alpha_l b_\alpha^j \rho^{\alpha-j-l} \bar{\partial} \rho_l \wedge \bar{\partial} \rho_j + \mathcal{O}^{k-1}(M).$$

A straightforward calculation yields

$$\frac{b_\alpha^j}{\alpha_j} = \frac{b_\alpha^l}{\alpha_l} + \mathcal{O}(M) \tag{2.1.2}$$

for all α_j and α_l which are not 0. However, by the definition of g_k , b_α^j appears in (2.1.1) only if $\alpha_j \neq 0$. Thus for each α with $|\alpha| = k$ define

$$b_\alpha \equiv \frac{b_\alpha^j}{\alpha_j} \pmod{\mathcal{O}(M)}, \tag{2.1.3}$$

where for each $j = 1, 2, \dots, m$, b_α^j appears in (2.1.1). The b_α 's are well defined by (2.1.2) and g_k defined by

$$g_k \equiv g - \sum_{|\alpha|=k} b_\alpha \rho^\alpha \tag{2.1.4}$$

is a member of ${}^k[f]$. \square

The following corollaries are immediate:

COROLLARY 2.2. *Let $f \in \mathcal{Q}^{(0,0)}$ and k be any positive integer. ${}^k[f] \neq \emptyset$ if and only if $\bar{\partial}_j^{j-1}[f] = 0$ for every $j = 1, 2, \dots, k$.*

COROLLARY 2.3. *For any positive integer k the condition*

$$\mathcal{C}_k: \bar{\partial}_j^{j-1}[f] = 0 \text{ for every } j = 1, 2, \dots, k$$

is biholomorphically invariant.

Theorem 2.1 and a well-known argument using some L^2 -estimates of L. Hormander imply a function theoretic result. (See Hormander and Wermer [6] or R. Nirenberg and Wells [7].)

THEOREM 2.4. *Suppose U is a domain of holomorphy and K is a compact subset of M which is holomorphically convex with respect to U . If $f \in \mathcal{O}^{(0,0)}$ and satisfies \mathcal{C}_k for every $k = 1, 2, \dots, \infty$ then f is the uniform limit on K of holomorphic functions on U .*

REMARK 2.5. If M is a C.R. submanifold then condition \mathcal{C}_1 implies conditions \mathcal{C}_k for all $k \geq 1$. Thus a C.R. function satisfies all conditions \mathcal{C}_k . Therefore Theorem 2.4 extends a previously known result to non-C.R. submanifolds. (See Freeman [2] and Hormander and Wermer [6].)

In [4] this author characterized the real-analytic functions which are locally the restrictions of ambient holomorphic functions to a real-analytic submanifold M , provided M is generic away from its C.R. singularities. Whereas Theorem 4.6 of [4] gives the characterization in terms of a local parameterization, the following theorem is in terms of defining functions for M .

THEOREM 2.6. *Suppose M is real-analytic. A real-analytic function f is the restriction (near 0) of an ambient holomorphic function if and only if f satisfies condition \mathcal{C}_k for all $k = 1, 2, \dots, \infty$.*

Theorem 2.6 is a consequence of Theorem 2.1 (in particular (2.1.4)) and the following lemma whose proof requires some independent machinery which the reader might wish to postpone until the end.

LEMMA 2.7. *Suppose M is real-analytic. Let $f \in \mathcal{O}^{(0,0)}$ be real-analytic (near 0) and suppose for every $k = 1, 2, \dots, \infty$ there exists a real-analytic function f_k (near 0) such that*

$$\left. \begin{aligned} \text{(a)} \quad & f = f_k + \mathcal{O}(M) \\ \text{(b)} \quad & \bar{\partial} f_k \in \mathcal{O}^k(M) \end{aligned} \right\} \text{ i.e. } f_k \in {}^k[f],$$

$$\text{(c)} \quad f_k = f_{k-1} + \mathcal{O}^k(M).$$

Then there is an ambient holomorphic function F (near 0) so that $f = F + \mathcal{O}(M)$.

PROOF. Let C_n , respectively R_k , denote the convergent power series centered at 0 with complex coefficients and n complex, respectively k real, indeterminates. Suppose $\Phi = (\phi_1, \dots, \phi_n)$ is a parameterization for M with $\Phi(0) = 0$. Thus $\phi_j \in R_k$ for all $j = 1, 2, \dots, n$. For any $f \in R_k$ let \tilde{f} denote its unique holomorphic extension to $C^k = R^k + iR^k$ and define $\tilde{\Phi} \equiv (\tilde{\phi}_1, \dots, \tilde{\phi}_n)$. Let ν denote the induced ring isomorphism between R_k and C_k .

Lemma 2.7 will be proved by constructing $F \in C_n$ such that $\tilde{f} \circ \tilde{\Phi} = F \circ \tilde{\Phi}$. (See [4] for a discussion of the sufficiency of such F .) Let \hat{C}_n ,

respectively \hat{R}_k , denote the completion in the Krull topology of C_n , respectively R_k (i.e. the formal power series rings). For $f \in \hat{R}_k$ define $\tilde{f} \in \hat{C}_k$ in the obvious manner. Let $F = \lim_{k \rightarrow \infty} f_k$ (limit in the Krull topology). It follows from (c) that $F \in \hat{R}_{2n}$ is well defined, and (b) implies that $\bar{\partial}F \equiv 0$. Thus $F \in \hat{C}_n$. Finally, (a) and the definition of "limit in Krull topology" imply $F \circ \Phi \equiv f \circ \Phi$. Because $F \in \hat{C}_n$ it follows that $F \circ \tilde{\Phi} = \tilde{f} \circ \tilde{\Phi}$. (ν induces an isomorphism between \hat{R}_k and \hat{C}_k .) Thus $F \circ \tilde{\Phi} \in C_k$. It is easy to see that the hypothesis " $E \cap M$ is nowhere dense in M " is equivalent to "the generic rank of $\tilde{\Phi}$ is n " (see §4 of [4]). Thus by Theorem 1.2 of Eakin and Harris [1] it follows that $F \in C_n$. \square

Applying Remark 2.5 yields a known result in case M is a C.R. submanifold (see Tomassini [8]). Moreover in this case one can relax the hypothesis that $E \cap M$ is nowhere dense in M by considering the minimal complex envelope (see [4]).

3. Application. Consider the problem of describing the conditions \mathcal{C}_k in terms of defining functions ρ_1, \dots, ρ_m for M . Without loss of generality assume Z_1, \dots, Z_n are holomorphic coordinates for C^n defined on an open set U containing 0 such that $\{Z \in U | \bar{\partial}_z \rho_1 \wedge \dots \wedge \bar{\partial}_z \rho_m \wedge d\bar{z}_{m+1} \wedge \dots \wedge d\bar{z}_n = 0\} = E$. (E is defined in §2 and is assumed to satisfy the previous hypothesis.) The following technical lemma will be useful.

LEMMA 3.1. *Suppose D_1, \dots, D_n are C^∞ vector fields of type (0, 1) defined on $U|E$ which satisfy*

- (a) $D_j(\rho_i) = \delta_i^j$ for all $i, j = 1, 2, \dots, m$,
- (b) $D_{m+j}(\bar{Z}_{m+i}) = \delta_i^j$ for all $i, j = 1, 2, \dots, n - m$,
- (c) $D_j(\bar{Z}_{m+i}) = 0$ and $D_{m+i}(\rho_j) = 0$ for all $i = 1, 2, \dots, m - n$ and $j = 1, 2, \dots, m$.

(δ_i^j is the Kronecker delta function.) $f \in \mathcal{O}^{(0,0)}$ satisfies \mathcal{C}_k if and only if for each $j = 1, 2, \dots, k$ and any $f_{j-1} \in j^{-1}[f]$

- (1) $D_1^{\alpha_1} \dots D_m^{\alpha_m} f_{j-1}|_M \in \mathcal{O}^{(0,0)}(M)$ for any $|\alpha| = j$, and
- (2) $D_{m+l}(f_{j-1}) \in \mathcal{O}(M|E)$ for all $l = 1, 2, \dots, n - m$.

REMARK 3.2. M is C.R. if and only if $E = \emptyset$ in which case (1) is vacuous and hence (2) is equivalent to the tangential Cauchy-Riemann equations.

PROOF OF LEMMA 3.1. "only if": Suppose f satisfies \mathcal{C}_k . Thus $1 < j < k$ and $f_{j-1} \in j^{-1}[f]$ implies by Corollary 2.2 and Theorem 2.1 that $\bar{\partial}f_{j-1} \in \mathcal{G}_{j-1}$. Let

$$\bar{\partial}f_{j-1} = \sum_{l=1}^m \sum_{\substack{|\alpha|=j-1 \\ \alpha_l \neq 0}} b'_\alpha \rho^{\alpha-l} \bar{\partial}\rho_l + \mathcal{O}^j(M). \tag{3.2.1}$$

Applying (a) to (3.2.1) j times yields

$$D_1^{\alpha_1} \dots D_m^{\alpha_m} (f_{j-1})|_M = \alpha! b_\alpha|_M \quad \text{for all } |\alpha| = j \tag{3.2.2}$$

(b_α defined by (2.1.3)). (3.2.2) implies (1) and (c) implies (2).

“If”: Suppose (1) and (2) are true, $1 < j \leq k$, and $f_{j-1} \in j^{-1}[f]$. Define $f_j \in \mathcal{C}^{(0,0)}$ via

$$f_j \equiv f_{j-1} - \sum_{|\alpha|=j} (1/\alpha!) D_1^{\alpha_1} \cdots D_m^{\alpha_m} f_{j-1}|_M \rho^\alpha. \tag{3.2.3}$$

It follows from (a), (b), and (c) that $D^\alpha f_j \in \mathcal{O}(M|E)$ for all $|\alpha| \leq j$. Thus by continuity and the hypothesis on E it follows that $\bar{\delta} f_j \in \mathcal{O}^j(M)$. But j was arbitrary between 1 and k , thus $k[f] \neq \emptyset$ and f satisfies \mathcal{C}_k by Corollary 2.2.

□

Vector fields satisfying (a), (b), and (c) of Lemma 3.1 can be constructed as follows. Let A be the $n \times n$ matrix defined by

$$A \equiv \begin{bmatrix} \frac{\partial \rho_i}{\partial \bar{z}_j} \\ \dots \dots \dots \\ 0 \vdots I_{n-m, n-m} \end{bmatrix}, \quad i = 1, 2, \dots, m, j = 1, 2, \dots, n.$$

Then $E = \{z \in U | \det A(z) = 0\}$. The vector fields defined on $U|E$ by

$$(D_1, \dots, D_n) \equiv (\partial/\partial \bar{z}_1, \dots, \partial/\partial \bar{z}_n) A^{-1} \tag{3.2.4}$$

satisfy (a), (b), and (c) of Lemma 3.1. Moreover (3.2.3) provides a scheme for making successively higher order corrections to f . Thus in the real-analytic setting (3.2.3) and Lemma 2.7 provide a construction of the holomorphic extension of f ; namely,

$$F \equiv f + \sum_{j=1}^{\infty} (-1)^j \sum_{|\alpha|=j} (1/\alpha!) D_1^{\alpha_1} \cdots D_m^{\alpha_m} f_{j-1}|_M \rho^\alpha \tag{3.2.5}$$

where f_{j-1} is any member of $j^{-1}[f]$.

As an example of the explicit calculation of the D_j 's and hence the \mathcal{C}_k 's, consider the case of a real 2-dimensional submanifold M contained in \mathbb{C}^2 with $E \cap M$ nowhere dense in M . Computing D_1 and D_2 using (3.2.4) yields

$$D_1 = \left(\frac{1}{D} \right) \left\{ \frac{\partial \rho_2}{\partial \bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho_2}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2} \right\}$$

and

$$D_2 = \left(\frac{1}{D} \right) \left\{ -\frac{\partial \rho_1}{\partial \bar{z}_2} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial \rho_1}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2} \right\}$$

where

$$D = \frac{\partial \rho_1}{\partial \bar{z}_1} \frac{\partial \rho_2}{\partial \bar{z}_2} - \frac{\partial \rho_1}{\partial \bar{z}_2} \frac{\partial \rho_2}{\partial \bar{z}_1}.$$

Without loss of generality assume $\rho_1 = (1/2)(z_2 + \bar{z}_2) - h(z_1)$ and $\rho_2 = (-i/2)(z_2 - \bar{z}_2) - g(z_1)$ for \mathcal{C}^∞ real valued functions h and g . If $G = h + ig$

then

$$\begin{aligned}
 D_1 &= \left(2 / \frac{\partial G}{\partial \bar{z}_1} \right) \left\{ -\frac{i}{2} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial g}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2} \right\}, \\
 D_2 &= \left(2 / \frac{\partial G}{\partial \bar{z}_1} \right) \left\{ -\frac{1}{2} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial h}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2} \right\}.
 \end{aligned}
 \tag{3.2.6}$$

For $f \in \mathcal{Q}^{(0,0)}$ it suffices to assume $\partial f / \partial \bar{z}_2 \equiv 0$ (i.e. restricting to M). From (3.2.6) for such f

$$D_1 f|_M = D_1 f = \left[-i / \frac{\partial G}{\partial \bar{z}_1} \right] \frac{\partial f}{\partial \bar{z}_1} = i D_2 f = i D_2 f|_M.$$

The condition \mathcal{C}_k is then equivalent to

$$D_1^j f \in \mathcal{Q}^{(0,0)} \quad \text{for all } j = 1, 2, \dots, k.$$

Consider the following examples, each of which was discussed in [4, §5].

EXAMPLE 3.3. (“CUP”) Let $\rho_1 = \frac{1}{2}(z_2 + \bar{z}_2) - z_1 \bar{z}_1$ and $\rho_2 = (-i/2)(z_2 - \bar{z}_2)$, then $D_1 = (i|z_1)\partial/\partial\bar{z}_1$ and $f \in \mathcal{Q}^{(0,0)}$ independent of z_2 , satisfies \mathcal{C}_k if and only if $(1|z_1^j)\partial^j f/\partial\bar{z}_1 \in \mathcal{Q}^{(0,0)}$ for all $j = 1, 2, \dots, k$. Thus if f is real-analytic then f has a holomorphic extension off M (near 0) if and only if $(\partial^{m+n} f / \partial z_1^m \partial \bar{z}_1^n)(0) = 0$ for all $m < n$.

EXAMPLE 3.4. (“SADDLE”) Let $\rho_1 = \frac{1}{2}(z_2 + \bar{z}_2) - \frac{1}{2}(z_1^2 + \bar{z}_1^2)$ and $\rho_2 = (-i/2)(z_2 - \bar{z}_2)$, then $D_1 = (i|\bar{z}_1)\partial/\partial\bar{z}_1$ and $f \in \mathcal{Q}^{(0,0)}$ (independent of z_2) satisfies \mathcal{C}_k if and only if $[(i|\bar{z}_1)\partial/\partial\bar{z}_1]^j(f) \in \mathcal{Q}^{(0,0)}$ for all $j = 1, 2, \dots, k$. For f real-analytic this implies f has a holomorphic extension off M (near 0) if and only if $(\partial^{m+n} f / \partial z_1^m \partial \bar{z}_1^n)(0) = 0$ for all pairs (m, n) with m odd.

EXAMPLE 3.5. (“PARABOLA”) Let $\rho_1 = \frac{1}{2}(z_2 + \bar{z}_2) - \frac{1}{4}(z_1 + \bar{z}_1)^2$ and $\rho_2 = (-i/2)(z_2 - \bar{z}_2)$, then $D_1 = (2i/(z_1 + \bar{z}_1))\partial/\partial\bar{z}_1$ and f satisfies \mathcal{C}_k if and only if $[(2i/(z_1 + \bar{z}_1))\partial/\partial\bar{z}_1]^j(f) \in \mathcal{Q}^{(0,0)}$ for all $j = 1, 2, \dots, k$.

In each of the above examples the conditions for holomorphic extendability of real-analytic functions are the same as those derived in [4, §5]. Whereas the conditions as derived in [4], a priori yield no information when applied to \mathcal{C}^∞ functions in general, it is now seen by Corollary 2.2 that these conditions do yield information when applied to such functions.

The relaxation of the hypothesis “ $E \cap M$ is nowhere dense in M ” appears to necessitate more study into the local geometry of a \mathcal{C}^∞ , or even real-analytic, submanifold in the vicinity of a C.R. singularity. (See [4, §6], for a discussion of this topic.)

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