EXTENSIONS RELATIVE TO A II\(_\infty\)-FACTOR

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Abstract. It will be shown that the equivalence classes of C*-algebra extensions of C(X) relative to a II\(_\infty\)-factor and Hom(K\(^4\)(X), \(\mathbb{R}\)) are isomorphic. This provides a proof for the result of Brown, Douglas and Fillmore [5] on the isomorphism between the former group and Hom(K\(^4\)(X), \(\mathbb{R}\)).

Let \(\mathcal{K}\) be a separable infinite dimensional Hilbert space, \(\mathcal{L}(\mathcal{K})\) the algebra of all bounded linear operators on \(\mathcal{K}\), \(\mathcal{K}(\mathcal{K})\) the ideal of compact operators, and \(\mathcal{A}(\mathcal{K})\) the quotient algebra \(\mathcal{L}(\mathcal{K})/\mathcal{K}(\mathcal{K})\). In [3], [4], [5], Ext \(X\) was defined as the set of equivalence classes of C*-algebra extensions,

\[0 \rightarrow \mathcal{K}(\mathcal{K}) \rightarrow \mathcal{E} \rightarrow C(X) \rightarrow 0\]

for \(X\) a compact metric space and \(C(X)\) the algebra of continuous complex-valued functions on \(X\), or equivalently as the unitary equivalence classes of unital *-monomorphism \(\tau: C(X) \rightarrow \mathcal{A}(\mathcal{K})\). It was shown that Ext \(X\) is an abelian group and that it is a generalized homology theory. One of the basic facts about Calkin algebra used in the theory of C*-algebra extensions is the Weyl-von Neumann Theorem, which was generalized to semifinite factors by L. Zsido [10]. P. A. Fillmore has extended part of this theory to semifinite factors [6]. That is, let \(M\) be a II\(_\infty\)-factor in \(\mathcal{K}\) with a dimension function \(d\), \(\mathcal{K}(M)\) the ideal generated by the finite projections, and \(\mathcal{A}(M)\) the quotient algebra \(M/\mathcal{K}(M)\). Ext\(^M\) \(X\) was defined as the set of unitary equivalence classes of unital *-monomorphisms \(\tau: C(X) \rightarrow \mathcal{A}(M)\). An extension \(\tau\) is trivial if \(\tau\) can be factored through \(M\), that is, if there exists a unital *-monomorphism \(\sigma\) of \(C(X)\) into \(M\) such that \(\tau = \pi \cdot \sigma\), where \(\pi\) is the natural homomorphism of \(M\) onto \(M/\mathcal{K}(M)\). The sum of \(\tau_1\) and \(\tau_2\) is the extension defined as follows: choose isometries \(V_1\) and \(V_2\) in \(M\) such that \(V_1V_1^* + V_2V_2^* = 1\), and let

\[(\tau_1 + \tau_2)(f) = \pi(V_1)\tau_1(f)\pi(V_1^*) + \pi(V_2)\tau_2(f)\pi(V_2^*)\]

for all \(f\) in \(C(X)\). The equivalence class of \(\tau_1 + \tau_2\) is independent of choice of isometries in the definition, trivial extensions form the identity element, and

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Ext\(^M X\) is an abelian group [6]. The techniques in [3] can be modified to prove that, for a closed subset \(A\) of \(X\) such that \(X/A\) is totally disconnected, the map \(\iota_\ast: \text{Ext}\_A X \rightarrow \text{Ext}\_M X\) induced by the inclusion map \(i: A \rightarrow X\) is an isomorphism. The rest of the proof for homotopy invariance in [5] goes through in our case.

Let \(\mathcal{C}\) be a separable C*-algebra. An extension for \(\mathcal{C}\) is a *-monomorphism of \(\mathcal{C}\) into \(\mathfrak{A}(M)\). Extension \(\tau_i: \mathcal{C} \rightarrow \mathfrak{A}(M), i = 1, 2\) are equivalent if there exists a partial isometry \(U\) in \(M\) such that \(\pi(U^*U)\) acts as a unit for \(\text{im } \tau_1\), \(\pi(UU^*)\) acts as a unit for \(\text{im } \tau_2\), and \(\tau_2(a) = \pi(U)\tau_1(a)\pi(U^*)\) for all \(a\) in \(\mathcal{C}\). By replacing the essential spectrum of an operator \(T\) in \(\mathfrak{L}(\mathcal{C})\) with the essential spectrum of an element \(T\) in \(M\) (i.e. the spectrum of \(\pi(T)\) in \(\mathfrak{A}(M)\)), in Lemma 3.3 of [5] and using the \(\Pi_\infty\)-factor version of [8] in Lemma 3.8, the argument in [5] can be modified to show that for any separable C*-algebra \(\mathcal{C}\) with Hausdorff spectrum \(X\) and a separable continuous field of Hilbert space \(E\) over \(X\) there is a natural bijection

\[
\text{Ext}^M \mathcal{C} \leftrightarrow \text{Ext}^M \left(\mathcal{C} \otimes_{\mathcal{C}(X)} E\right).
\]

\(E\) is the C*-algebra obtained from \(E\). In particular, \(\text{Ext}^M X\) is a \(K(X)\)-module.

Let \(\text{Ext}^r_M X = \text{Ext}^M(S^rX), r < 1\), where \(S^rX\) is the \(n\)th iterated suspension of \(X\). Since \(\text{Ext}^M X/A\) and \(\text{Ext}^M X\) are isomorphic for a contractible closed subset \(A\) of \(X\) (as in [5, Lemma 2.17]), the use of \(S^rX\) for both the reduced and unreduced suspensions will not cause confusion. For a closed subset \(A\) of \(X\) we have a one-sided long exact sequence

\[
\text{Ext}^r_M A \rightarrow \text{Ext}^r_M X \rightarrow \text{Ext}^r_M X/A \rightarrow \text{Ext}^r_M A \rightarrow \cdots.
\]

As in [5], by the same proof, there exist isomorphisms

\[
\text{Per}_r: \text{Ext}^r_M X \rightarrow \text{Ext}^r_M X, \quad \text{for } r < 1,
\]

hence \(\text{Ext}^M\) is a generalized homology theory.

Also there is a natural transformation \(\gamma_\infty: \text{Ext}^M X \rightarrow \text{Hom}(\tilde{K}(SX), R)\). More precisely, for a given unital *-monomorphism \(\tau\) of \(C(X)\) into \(\mathfrak{A}(M)\) and a continuous function \(f: X \rightarrow GL_n\), where \(GL_n\) is the group of invertible \(n \times n\) matrices, the desired map \(\gamma_\infty\) is induced by \((f, \tau) \rightarrow \text{ind}(\tau_n(f))\) where

\[
\tau_n(f) = (\tau(f_{ij})) \in \mathfrak{A}(M) \otimes M_n(C) \approx \mathfrak{A}(M \otimes M_n(C)) \approx \mathfrak{A}(M).
\]

Using the index theory of M. Breuer [1], [2] and the fact that \(\tilde{K}(SX) = \text{inj lim}[X, GL_n]\), it can be seen that this construction depends only on the homotopy class of \(f\) and the equivalence class of \(\tau\), it respects the obvious inclusion of \(GL_n\) into \(GL_{n+1}\), and it is a homomorphism.

Since \(R\) is an injective \(\mathbb{Z}\)-module, \(\text{Hom}(\tilde{K}(X), R)\) is a generalized homology theory. We will prove that this natural transformation \(\gamma_\infty\) is an isomorphism by use of standard techniques in algebraic topology. Both homologies satisfy two additional axioms introduced by Milnor [7] (such homology is called

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Steenrod homology). These are a strong excision axiom—which is automatic since we define $\text{Ext}^M(X, A) = \text{Ext}^M(X/A)$ and $\tilde{K}(X, A) = \tilde{K}(X/A)$—and the following cluster axiom. We say that a compact metric space $X$ is the strong wedge of its closed subspaces $X_1, X_2, \ldots$ if $X = \bigcup_{n=1}^{\infty} X_n$, diam $X_n \to 0$, and there exists a point $b$ in $X$ such that $X_m \cap X_n = \{b\}$ for all $m \neq n$. The cluster axiom for a homology $H$ reads: Let $X$ be the strong wedge of subspaces $X_1, X_2, \ldots$ and let $r_n: X \to X_n$ be the retraction carrying $X_j$ to $\{b\}$ for $j \neq n$. Then $a \to (r_1(a), r_2(a), \ldots)$ defines an isomorphism of $H_q(X)$ onto $\prod_{n=1}^{\infty} H_q(X_n)$ for all $q$. That $\text{Ext}^M$ satisfies the cluster axiom follows from the argument of [5]; the following well-known facts imply the cluster axiom for $\text{Hom}(\tilde{K}(\cdot), R)$.

**Lemma 1** [9]. $\tilde{K}(\cdot)$ is a continuous functor, i.e. for an inverse system $\{X_a, f_{ba}\}$ of compact Hausdorff spaces the canonical map:

$$\tilde{K}(\text{proj lim } X_a) \to \text{inj lim } \tilde{K}(X_a)$$

is an isomorphism.

**Lemma 2** ([11, Proposition 6, AII, p. 92]). For any direct system $\{E_a, f_{ba}\}$ of abelian groups the canonical map $u \to u \circ f$ is an isomorphism:

$$\text{Hom}(\text{inj lim } E_a, R) \to \text{proj lim } \text{Hom}(E_a, R)$$

where $f_a$ is the inclusion of $E_a$ into inj lim $E_a$.

An important consequence of the axioms for Steenrod homology theory $H$ is the following short exact sequence

$$0 \to \text{proj lim}^1 H_{q+1}(X_n) \to H_q(\text{proj lim } X_n) \xrightarrow{P} \text{proj lim} H_q(X_n) \to 0$$

where $P$ is the canonical homomorphism and proj lim$^1$ is the first derived functor of inverse limit. We recall that for an inverse system $\{A_i, P_i\}$ of abelian groups proj lim$^1 A_i$ is the cokernel of the homomorphism $\Pi A_i \to \Pi A_i$ defined by

$$(a_1, a_2, a_3, \ldots) \to (a_1 - p_1(a_2), a_2 - p_2(a_3), \ldots).$$

Notice that in proj lim$^1$ sequences for $\text{Hom}(\tilde{K}(X), R)$, $P$ is an isomorphism by virtue of Lemmas 1 and 2: i.e. $\text{Hom}(\tilde{K}(\cdot), R)$ is a continuous functor.

**Theorem.** The natural transformation $\gamma_\infty: \text{Ext}^M X \to \text{Hom}(\tilde{K}(SX), R)$ is an isomorphism for any compact metric space $X$.

**Proof.** The space $X$ can be identified as the inverse limit of finite complexes $X_n$ (see [7]). Since $\gamma_\infty$ (point) is an isomorphism, the maps $\gamma_\infty(X_n)$ are isomorphisms using the five lemma and the fact that both functors are finitely additive. Hence

$$\text{proj lim}^M \text{Ext}^M X_n \cong \text{proj lim} \text{Hom}(\tilde{K}(SX_n), R)$$

and

$$\text{proj lim}^1 \text{Ext}^M X_n \cong \text{proj lim}^1 \text{Hom}(\tilde{K}(SX_n), R).$$
The following commutative diagram of proj \( \lim^{(1)} \) sequences and the five lemma complete the proof:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{proj} \lim^{(1)} \text{Ext}^M_{q+1} X_n & \rightarrow & \text{Ext}^M_X & \rightarrow & \text{proj} \lim \text{Ext}^M_X & \rightarrow & 0 \\
& & \downarrow \gamma & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{proj} \lim^{(1)} \text{Hom} \left( \tilde{K}^{q+1}(SX_n), \mathbb{R} \right) & \rightarrow & \text{Hom} \left( \tilde{K}^q(SX), \mathbb{R} \right) & \rightarrow & \text{proj} \lim \text{Hom} \left( \tilde{K}^q(SX_n), \mathbb{R} \right) & \rightarrow & 0
\end{array}
\]

**Corollary 1.** The canonical map \( P : \text{Ext}^M_X \rightarrow \text{proj} \lim \text{Ext}^M X_n \) is an isomorphism, where \( X = \text{proj} \lim X_n \); that is, \( \text{Ext}^M \) is a generalized Steenrod homology theory which is also continuous.

**Corollary 2.** \( \text{Ext}^M X \) is independent of \( M \) and is a real vector space.

Since \( \text{Hom}_Z \left( \tilde{K}(X), \mathbb{R} \right) = \text{Hom}_Z \left( \tilde{K}(X), \text{Hom}_R(\mathbb{R}, \mathbb{R}) \right) \approx \text{Hom}_R(\tilde{K}(X) \otimes_Z \mathbb{R}, \mathbb{R}) \), we get the following (for the last isomorphism see §4 of All, [11]).

**Corollary 3.** Let \( \beta \) be the isomorphism: \( \text{Hom}_Z(\tilde{K}(X), \mathbb{R}) \rightarrow \text{Hom}_R(\tilde{K}(X) \otimes_Z \mathbb{R}, \mathbb{R}) \). Then the map \( \beta \circ \gamma_\infty : \text{Ext}^M X \rightarrow \text{Hom}_R(\tilde{K}^I(X), \mathbb{R}) \) is an isomorphism, where \( \tilde{K}^I(X) = \tilde{K}(X) \otimes_Z \mathbb{R} \).

**References**


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