A CHARACTERIZATION FOR THE PRODUCT OF CLOSED IMAGES OF METRIC SPACES TO BE A k-SPACE

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Abstract. We give, under [CH], a necessary and sufficient condition for the product of two closed images of metric spaces to be a k-space.

1. Introduction. In [14, Theorem 4.3], we proved the following result. Recall that a space \( X \) is said to belong to class \( \mathcal{K} \) if it is the union of countably many closed and locally compact subsets \( X_n \) such that \( F \subset X \) is closed whenever \( F \cap X_n \) is closed for all \( n \).

Theorem 1.0. Let \( X \) and \( Y \) be closed s-images of metric spaces. Then \( X \times Y \) is a k-space if and only if one of the following three properties holds:

1. \( X \) and \( Y \) are metrizable spaces.
2. \( X \) or \( Y \) is a locally compact, metrizable space.
3. \( X \) and \( Y \) are spaces of class \( \mathcal{K} \).

In that place, we raised the question whether this theorem remains true if “s-images” is weakened to “images”.

In this paper, under the continuum hypothesis [CH], we shall give the following affirmative answer to this question.

Theorem 1.1 [CH]. Let \( X \) and \( Y \) be closed images of metric spaces under maps \( f \) and \( g \) respectively. Then \( X \times Y \) is a k-space (equivalently, \( f \times g \) is a quotient map by [6, Theorem 1.5]) if and only if one of the three properties of Theorem 1.0 holds.

Throughout this paper, we shall assume that all spaces are regular \( T_2 \), and all maps are continuous surjections.

2. Preliminaries. A space \( X \) is Fréchet if, whenever \( x \in \overline{A} \), then some sequence of points of \( A \) converges to \( x \). Obviously, every closed image of a first countable space is Fréchet.

Recall that a space \( X \) is strongly Fréchet [10] (= countably bi-sequential in the sense of E. Michael [7]) if, whenever \( \{ F_n; n = 1, 2, \ldots \} \) is a decreasing sequence accumulating at \( x \in X \), there exist \( x_n \in F_n \) such that the sequence \( \{ x_n; n = 1, 2, \ldots \} \) converges to \( x \). Clearly every strongly Fréchet space is Fréchet.
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Lemma 2.1 (cf. [7, Theorem 9.9]). Let $X$ be the closed image of a metric space (more generally, paracompact space) under a map $f$. If $X$ is strongly Fréchet, then $\partial f^{-1}(x)$ is compact for every $x \in X$.

Since every Fréchet space is a sequential space, by [13, Lemma 2.1 (A) and Proposition 2.4] and [12, Theorem 2.2], we have

Lemma 2.2. Let $X$ be a Fréchet space, and let $Y$ be a metric space. Suppose that $X \times Y$ is a k-space. Then $X$ is strongly Fréchet, or $Y$ is locally compact.

Lemma 2.3. Let $X$ be a Fréchet space, or a k-space each of whose points is a G$_\delta$-set. Let $Y$ be the closed image of a collectionwise normal and Fréchet space $Z$ under a map $f$. Suppose that $X \times Y$ is a k-space. Then $X$ is strongly Fréchet, or every $\partial f^{-1}(y)$ has property (P) below.

(P) Every subset of cardinality $2^{<\kappa}$ in $\partial f^{-1}(y)$ has an accumulation point.

Proof. Suppose that there is $y_0 \in Y$ such that $\partial f^{-1}(y_0)$ does not have property (P). Then there is a closed discrete subset $\{x_\alpha; \alpha \in A\}$ of $\partial f^{-1}(y_0)$ with $|A| = 2^{<\kappa}$. Since $Z$ is collectionwise normal, there is a discrete open collection $\{U_\alpha; \alpha \in A\}$ in $Z$ with $x_\alpha \in U_\alpha$. Since $Z$ is Fréchet, and $x_\alpha \in U_\alpha - f^{-1}(y_0)$ for each $\alpha \in A$, then there is a convergent sequence $\{x_{\alpha_1}; i = 1, 2, \ldots\}$ of $U_\alpha - f^{-1}(y_0)$ with its limit point $x_\alpha$. Let $C_\alpha = \{x_{\alpha_1}; i = 1, 2, \ldots\} \cup \{x_\alpha\}$ for each $\alpha \in A$, and let $Z_0 = \bigcup_{\alpha \in A} C_\alpha$. Then, since $\{C_\alpha; \alpha \in A\}$ is a discrete closed collection in $Z$, $Z_0$ is a closed subset of $Z$. Let $g = f|Z_0$. Then $g$ is a closed map from the locally compact, metric space $Z_0$. Let $Y_1 = \{y \in Y_0; g^{-1}(y)$ is not compact\}, where $Y_0 = g(Z_0)$. Then, by [8, Theorem 4], $Y_1$ is a closed discrete subset of $Y_0$. It is easy to see that $y_0 \in Y_1$. Since the sequence $g(C_\alpha)$ converges to $y_0$, and $Y_1$ is closed and discrete, then each $C_\alpha$ intersects only a finite number of $g^{-1}(y)$, $y \in Y_1$. Hence $C'_\alpha = C_\alpha - g^{-1}(Y_1)$ is infinite, which implies that each sequence $C'_\alpha$ converges to $x_\alpha$. For each $\alpha \in A$, let $A_\alpha = g(C'_\alpha)$. Then $\mathcal{A} = \{A_\alpha; \alpha \in A\}$ is locally finite, hence point-finite in $Y_0 - Y_1$. For, $g$ is a perfect map on $Z'_0 = Z_0 - g^{-1}(Y_1)$ and $\{C'_\alpha; \alpha \in A\}$ is a discrete collection in $Z'_0$. Since each $A_\alpha$ is countable, for each $\alpha \in A$, $A(\alpha) = \{\beta \in A; A_\alpha \cap A_\beta \neq \emptyset\}$ is at most countable. Then, there is a subset $A'$ of $A$ with cardinality $2^{<\kappa}$, such that $\mathcal{A}' = \{A_\alpha; \alpha \in A'\}$ is pairwise disjoint. Indeed, let $A = \{\alpha; \alpha < 2^{<\kappa}\}$. Then, for each $\alpha$, there is a pairwise disjoint subcollection $\mathcal{B}_\alpha$ of $\mathcal{A}$ such that $|\mathcal{B}_\alpha| < |\alpha|$ and $\bigcup_{\beta < \alpha} \mathcal{B}_\beta \subset \mathcal{B}_\alpha$. For, let $\{\mathcal{B}_\beta; \beta < \alpha\}$ be defined for each $\beta < \alpha$. Then we can choose $A_\alpha' \in \mathcal{A}$ with

$$A_\alpha' \cap \left( \bigcup_{\beta < \alpha} \{A_\delta; \delta \in \mathcal{B}_\beta\} \right) = \emptyset,$$

for each $A(\delta)$ is at most countable and $|\bigcup_{\beta < \alpha} \mathcal{B}_\beta| < |\alpha|$ ($\neq 2^{<\kappa}$). Let $\mathcal{B}_a = \{A_\alpha'\} \cup \bigcup_{\beta < \alpha} \mathcal{B}_\beta$. Then $\mathcal{B}_a$ satisfies the conditions. Hence, $\mathcal{A}' = \bigcup_{\alpha < 2^{<\kappa}} \mathcal{B}_a$ is a pairwise disjoint subcollection of $\mathcal{A}$ with cardinality $2^{<\kappa}$. 

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Now, let $Z_1 = \bigcup_{x \in A'} \{ C'_x \cup \{ x_a \} \}$. Let $h = f|Z_1$. Then, since $Z_1$ is closed in $Z$, $h$ is a closed map, hence is quotient. Moreover, $h(x_a) = y_0$ for each $\alpha \in A'$ and $h$ is one-to-one on $\bigcup_{\alpha \in A'} C'_\alpha$ by the choice of the index set $A'$. Here, we may assume that $h|C'_\alpha$ is one-to-one for each $\alpha \in A'$. Thus, $h(Z_1)$ can be shown to be homeomorphic to a quotient space $Z_1/F_1$ obtained from $Z_1$ identifying all points of $F_1 = h^{-1}(y_0)$.

On the other hand, $X \times h(Z_1)$ is a closed subset of a $k$-space $X \times Y$, for $h(Z_1)$ is closed in $Y$. Hence $X \times h(Z_1)$ is a $k$-space. This implies that $X \times (Z_1/F_1)$, which is homeomorphic to $X \times h(Z_1)$, is a $k$-space. Thus, by [15, Lemma 2.1(2)], $X$ is strongly Fréchet or $\partial Z_1$ has property (P). However, $\partial Z_1/F_1$ contains a closed discrete subset $\{ x_\alpha; \alpha \in A' \}$ of cardinality $2^\alpha$. Then it does not have property (P). Therefore $X$ is strongly Fréchet. That completes the proof.

**Proposition 2.4** [CH]. Let $X$ be a Fréchet space, or a $k$-space each of whose points is a $G_\delta$-set. Let $Y$ be the closed image of a first countable, paracompact space under a map $f$. If $X \times Y$ is a $k$-space, then either $X$ is strongly Fréchet, or $\partial f^{-1}(y)$ is locally compact and Lindelöf for every $y \in Y$.

**Proof.** Suppose that $X$ is not strongly Fréchet. Then, without [CH], every $\partial f^{-1}(y)$ is locally compact by [15, Theorem 2.2]. Moreover, from Lemma 2.3, every $\partial f^{-1}(y)$ has property (P). Then, under [CH] it is easy to see that every $\partial f^{-1}(y)$ is Lindelöf, for every $\partial f^{-1}(y)$ is paracompact.

**3. Proof of Theorem 1.1 and a related result.**

**Proof of Theorem 1.1.** The “if” part is that of Theorem 1.0 stated in §1. So we shall prove the “only if” part.

(i) Suppose that every $\partial f^{-1}(x)$ is Lindelöf: If every $\partial g^{-1}(y)$ is also Lindelöf, as in the proof of [5, Corollary 1.2], we may assume that $X$ and $Y$ are closed $s$-images of metric spaces. Thus, by the “only if” part of Theorem 1.0, the assertion holds. If some $\partial g^{-1}(y_0)$ is not Lindelöf, then $X$ is strongly Fréchet by Proposition 2.4. Thus $X$ is metrizable by Lemma 2.1. On the other hand, $Y$ is not strongly Fréchet by Lemma 2.1, for $\partial g^{-1}(y_0)$ is not compact. Hence $X$ is locally compact by Lemma 2.2.

(ii) Suppose that some $\partial f^{-1}(x_0)$ is not Lindelöf: Then, as above, $Y$ is locally compact and metrizable. That completes the proof.

As for the product of closed images of locally compact metric spaces, we have the following theorem, which is an improvement of [15, Proposition 2.6 or 2.7]. The “only if” part follows from the proof of Theorem 1.1. The “if” part follows from Proposition 3.2 below.

**Theorem 3.1** [CH]. Let $f_i : X_i \to Y_i$ ($i = 1, 2$) be closed maps such that each $X_i$ is a locally compact metric space (more generally, locally compact, Fréchet and paracompact space). Then $Y_1 \times Y_2$ is a $k$-space if and only if either of the following properties holds:
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(1) Every \( \partial f_{1}^{-1}(y_{1}) \) is compact, or every \( \partial f_{2}^{-1}(y_{2}) \) is compact. (Hence, \( Y_{1} \) or \( Y_{2} \) is locally compact.)

(2) Every \( \partial f_{i}^{-1}(y_{i}) \) is Lindelöf for \( i = 1, 2 \).

Proposition 3.2. (a) [4, Theorem 3.2] Let \( Y_{1} \) be a k-space, and let \( Y_{2} \) be a locally compact space. Then \( Y_{1} \times Y_{2} \) is a k-space.

(b) [15, Lemma 2.5] Let \( Y_{i} \) (\( i = 1, 2 \)) be closed images of locally compact spaces under maps \( f_{i} \) with each \( \partial f_{i}^{-1}(y_{i}) \) Lindelöf. Then \( Y_{1} \times Y_{2} \) is a k-space.

4. Some remarks to Theorem 1.1.

Remark 4.1. Concerning the “Fréchetness” for the product of two closed images of metric spaces, we have the following theorem from [9, Theorem 9.2] (also cf. [7, Proposition 4.D.5]), together with Lemma 2.1.

Theorem. Let \( X \) and \( Y \) be closed images of metric spaces. Then \( X \times Y \) is a Fréchet space (equivalently, hereditary k-space by [2]) if and only if either of the following properties holds:

1. \( X \) and \( Y \) are metrizable spaces.
2. \( X \) or \( Y \) is a discrete space.

Remark 4.2. Concerning the “\( k \)-ness” for the product of countably many copies of a closed image of a metric space, we have the following theorem from [13, Theorem 1.3] and [7, Theorem 7.3].

Theorem. Let \( X \) be a closed image of a metric space. Then \( X^{\omega} \) is a k-space if and only if \( X \) is a metrizable space.

Remark 4.3. As generalizations of metric spaces, J. G. Ceder [3] introduced three types of topological spaces which he called \( M_{1}, M_{2} \) and \( M_{3} \)-spaces, and observed that \( M_{1} \Rightarrow M_{2} \Rightarrow M_{3} \). An \( M_{1} \)-space is a regular space having a \( \sigma \)-closure preserving base. That every closed image of a metric space is \( M_{1} \) was proved by F. Slaughter [11]. The following example shows that Theorem 1.1 becomes false if “closed images of metric spaces” is weakened to “\( M_{1} \)-spaces”, even if in property (1) of Theorem 1.0 we replace “metrizable spaces” by “first countable spaces”.

Example. Let \( X \) be the Nagata space constructed in Example 9.2 in [3] \((X = \{(x, y); 0 < x < 1, y > 0\})\): the topology on \( X \) has a base consisting of disks missing the x-axis and sets of the form \( U_{n}(p) = \{p\} \cup \{(x, y); |x - p| < 1/n \text{ and } y \text{ lies below the graph of } (x - p)^{2} + (y - n)^{2} = n^{2}\}\). Obviously \( X \) is separable, first countable and not second countable. Hence \( X \) is not metrizable. The proof that \( X \) is \( M_{1} \), which is due to J. Nagata, is given in [3].

Let \( C \) be a closed interval contained in \((0, 1)\). Let \( Y \) be a quotient space obtained by identifying all points of \( C \), and let \( f: X \to Y \) be the natural quotient map. Since \( C \) is compact in \( X \), \( f \) is a perfect map. Then \( Y \times Y \) is a k-space, for it is the perfect image of a first countable space \( X \times X \). To show that \( Y \) is \( M_{1} \), let \( \mathcal{B} = \bigcup_{i=1}^{\omega} \mathcal{B}_{i} \) be a \( \sigma \)-closure preserving base for \( X \). We may assume that \( \mathcal{B}_{i} \subset \mathcal{B}_{i+1} \) for each \( i \), and that each \( \mathcal{B}_{i} \) is closed under arbitrary
unions. Then, since $C$ is compact in $X$, \{ $f(B); \ B \in \mathfrak{B}$ with $C \subset B$ or $C \cap B = \emptyset$ \} is a $\alpha$-closure preserving base for $Y$.

That $Y$ is not first countable will be shown below, hence neither is $Y$ locally compact by [3, Corollary 5.7]. Suppose that $Y$ is first countable. Then the compact, separable metric subset $C$ is of countable character in $X$ (Arhangel’skiï [1, Definition 3.5]). Then, by [1, Lemma 3.2], there is a countable collection $\mathfrak{B}$ of open subsets of $X$ such that, if $c \in C$ and $c \in U$ with $U$ open in $X$, then $c \in V \subset U$ for some $V \in \mathfrak{B}$. This implies that a subspace $C \times \{ y; y > 0 \}$ of $X$ is second countable. But this is a contradiction, for the subspace is obviously non-second countable.

To show that $Y$ is not a space of class $\mathfrak{L}'$, suppose not. Then $Y$ is the union of countably many closed and locally compact subsets $Y_n$ such that, $F \subset Y$ is closed whenever $F \cap Y_n$ is closed for each $n$. We may assume that $Y_n \subset Y_{n+1}$ for each $n$. Then each compact subset of $Y$ is contained in some $Y_n$. For any $x \in X$, let $\{ V_n; n = 1, 2, \ldots \}$ be a decreasing local base at $x$. Then, for some $m$, $f(V_m) \subset Y_m$, hence $V_m \subset f^{-1}(Y_m)$. While, since $f$ is perfect, $f^{-1}(Y_m)$ is locally compact. Hence, by [3, Corollary 5.7], $f^{-1}(Y_m)$ is metrizable, so is $V_m$. This implies that $X$ is a locally metrizable space. Then $X$ is metrizable, for it is Lindelöf. But, this is a contradiction to the fact that $X$ is nonmetrizable. Thus $Y$ is not a space of class $\mathfrak{L}'$.

REFERENCES


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