PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 1, April 1979

MORE PARACOMPACT BOX PRODUCTS

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ABSTRACT. We show that if there is no family of cardinality less than $\mathfrak{c}$ which dominates ${}^{\omega}\omega$, then the box product of countably many compact first-countable spaces is paracompact; hence the countable box product of compact metrizable spaces is paracompact if $2^{\omega} = \omega_2$. We also give classes of forcing extensions in which many box products are paracompact.

0. Introduction. In the last six years there have been many proofs showing that various classes of countable box products were consistently paracompact. The methods of these proofs are two: the tree argument used by van Douwen, and the stratification argument used by everyone else. Here we generalize the latter technique to show:

THEOREM 0. (a) *If no family of cardinality less than* $\mathfrak{c}$ *dominates* ${}^{\omega}\omega$, *then the box product of countably many compact first-countable spaces is paracompact.*

(b) *If* $\mathfrak{c} = \omega_2$, *then the box product of countably many compact metrizable spaces is paracompact.*

We then generalize the method further to show that a simplified version of the principle implicit in **[Ro]** also implies that the box product of countably many compact first-countable spaces is paracompact. From this we find a wide class of forcing extensions in which various box products are paracompact, thus pointing out where not to look in trying to give a negative answer to the following open questions:

A. Is $\square^{\omega}(\omega + 1)$ paracompact?

B. Does the existence of a λ-scale for some λ imply that the countable box product of compact first-countable spaces is paracompact?

Thanks are due to William Mitchell and Saharon Shelah who pointed out that the proofs in this paper did more than they were expected to do.

1. Preliminaries. All spaces are assumed to be Hausdorff. A space X is paracompact iff every open covering of X has a locally finite refinement. The letters f, g, h are reserved for functions from ω to ω; A, B for infinite subsets of ω.

Let I be an index set, X_i a topological space for each $i \in I$. Then the box product $\square_{i\in I} X_i$ consists of all points in $\prod_{i \in I} X_i$ under the topology whose

Received by the editors February 6, 1978 and, in revised form, May 1, 1978.
AMS (MOS) subject classifications (1970). Primary 54D20; Secondary 04A20.

basis consists of all boxes $u = \prod_{i \in I} u_i$, where each u_i is open in X_i. If $x = \langle x_i: i < \omega \rangle$ and $y = \langle y_i: i < \omega \rangle$ are in $\square_{i \in \omega} X_i$, we say $x \equiv y$ iff $\{i: y_i \neq x_i\}$ is finite. $\nabla_{i \in \omega} X_i$ is the quotient topology on the equivalence classes of $\square_{i \in \omega} X_i$ generated by $\equiv$.

The connection between $\square$ and ∇, and the reasons ∇ is nice to work with, are made clear by the following theorem of Kunen:

THEOREM 1 (KUNEN). (a) *If each X_i is compact, then $\square_{i \in \omega} X_i$ is paracompact iff $\nabla_{i \in \omega} X_i$ is paracompact.*

(b) *G_δ's in any $\nabla_{i \in \omega} X_i$ are open; hence if each X_i is regular, the G_δ's of $\nabla_{i \in \omega} X_i$ are clopen, and the space is 0-dimensional.*

(c) *$\nabla_{i \in \omega} X_i$ is paracompact iff every open cover has an open disjoint covering refinement (such a space is called ultraparacompact).*

Because of Kunen's theorem, for the rest of this paper we work with ∇, and will prove that under various circumstances $\nabla_{i \in \omega} X_i$ is paracompact, where each X_i is regular. The reader can then conclude that if each X_i satisfies the desired hypotheses and is also compact, then $\square_{i \in \omega} X_i$ is paracompact as well.

The connection between ∇ and the structure of ${}^{\omega}\omega$ is made clear by the following notation and definitions.

Suppose, for each $i \in \omega$, X_i is first countable. For each $x_i \in X_i$, we fix u_{x_i} a function from ω such that $\{u_{x_i}(j): j \in \omega\}$ is a descending neighborhood basis for x_i. Then given $f: \omega \to \omega$ and $x \in \nabla_{i \in \omega} X_i$, we define $u_{x,f} = \nabla_{i \in \omega} u_{x_i}(f(i))$.

Given f, g, A, we say $f < g$ on A iff $\{i \in A: f(i) \not< g(i)\}$ is finite. Then the following fact is immediate:

FACT 2. If $x, y \in \nabla_{i \in \omega} X_i$ and $f, g \in {}^{\omega}\omega$, then letting $C = \{i: u_{x_i}(f(i)) \cap u_{y_i}(g(i)) = \varnothing\}$, C is infinite iff $u_{x,f} \cap u_{y,g} = \varnothing$; and if $u_{x,f} \cap u_{y,g} = \varnothing$ and $h \nleq f$ on C, then $u_{x,h} \cap u_{y,g} = \varnothing$.

The careful reader will observe in the preceding sentence a slight confusion of events in $\prod X_i$ and ∇X_i. Such confusions will continue to occur for the sake of readability.

$\mathfrak{F} \subset {}^{\omega}\omega$ is a dominating family iff for every $f \in {}^{\omega}\omega$ there is some $g \in \mathfrak{F}$, $g < f$ on ω. A λ-scale is a dominating family well ordered by $<$ of order type λ. The space $\omega + 1$ is the order type $\omega + 1$ under the interval topology.

We write $\mathfrak{D}_\alpha \nearrow \mathfrak{D}$, $\alpha < \lambda$, iff $\mathfrak{D} = \bigcup_{\alpha < \lambda} \mathfrak{D}_\alpha$ and $\alpha < \beta \Rightarrow \mathfrak{D}_\alpha \subseteq \mathfrak{D}_\beta$.

2. Combinatorial proofs. In this section we state two combinatorial principles and prove that they establish paracompactness.

DEFINITION 3. Let $\mathfrak{F} \subset {}^{\omega}\omega$, $\mathfrak{A} \subset \mathfrak{P}(\omega)$. We say $\mathfrak{F}$ is cofinal on $\mathfrak{A}$ iff for every f there exists an $A \in \mathfrak{A}$ and a $g \in \mathfrak{F}$ such that $g > f$ on A.

The principle $(*)$ says:

$$\text{If } |\mathfrak{F}|, |\mathfrak{A}| < \mathbf{c}, \text{ then } \mathfrak{F} \text{ is not cofinal on } \mathfrak{A}.$$

FACT 4. $(*)$ is equivalent to the assertion that there is no dominating family of cardinality $< \mathbf{c}$.

PROOF. One direction is trivial. We prove the not-quite-so-trivial direction. Suppose (*) is false via $\mathfrak{F}$, $\mathfrak{A}$ and we have some $g \in {}^{\omega}\omega$. We may assume that each function in $\mathfrak{F}$ is increasing. There exists an $f \in \mathfrak{F}$, and an $A \in \mathfrak{A}$, with $f > g$ on A. Suppose $A = \{a_0, a_1, \dots\}$ in increasing order. Define h to be constantly equal to $f(a_{n+1})$ on the interval $[a_n, a_{n+1})$. Then $h > g$ on ω and there can be at most $|\mathfrak{F}| \cdot |\mathfrak{A}| < \mathbf{c}$ such h's.

Notes on (*). 1. (*) $\Rightarrow$ there is no λ-scale, for $\lambda < \mathbf{c}$.

2. By Fact 4, (*) is true if there is a $\mathbf{c}$-scale and $\mathbf{c}$ is regular, hence under MA.

By Fact 4 and Theorem 1, Theorem 0 (a) will be proved by the following

THEOREM 5. (*) *implies that* $\nabla_{i\in\omega} X_i$ *is paracompact if each* X_i *is first countable, Lindelöf, and regular.*

PROOF. By a well-known theorem of Arhangel'skiĭ, each X_i has cardinality $\leq \mathbf{c}$, hence $X = \nabla_{i\in\omega} X_i$ has cardinality $\mathbf{c}$. We construct a refinement of a given cover $\mathfrak{U}$ by an induction of length $\mathbf{c}$. Suppose we have assigned X some well ordering of type $\mathbf{c}$ and have already covered the first β elements of X by clopen sets, where this covering $\mathfrak{U}_\beta$ refines $\mathfrak{U}$, $|\mathfrak{U}_\beta| \leq |\beta| < \mathbf{c}$, and each $v \in \mathfrak{U}_\beta$ is of the form $\bigcap_{i\in\omega} u_{y,f_i}$, where $\{f_i: i \in \omega\}$ is an increasing sequence. Let x be the next point we have to cover. If x is already covered, we simply let $\mathfrak{U}_{\beta+1} = \mathfrak{U}_\beta$. Otherwise, for each $v \in \mathfrak{U}_\beta$ we can find a g_v so that $u_{x,g_v} \cap v = \varnothing$. Hence, where $v = \bigcap_{i\in\omega} u_{y,f_i}$, we can find an i_v so that

$$u_{x,g_v} \cap u_{y,f_{i_v}} = \varnothing$$

(this is because the f_i's are increasing). Hence by Fact 2,

$$A_v = \{k: u_{x_k}(g_v(k)) \cap u_{y_k}(f_{i_v}(k)) = \varnothing\}$$

is infinite. By hypothesis we can find a single h so that for every $v \in \mathfrak{U}_\beta$, $g_v \not< h$ on A_v. Hence by Fact 2, $u_{x,h} \cap v = \varnothing$ for all $v \in \mathfrak{U}_\beta$. Let h_i be an increasing sequence, $h_0 = h$. We let $\mathfrak{U}_{\beta+1} = \mathfrak{U}_\beta \cup \{\bigcap_{i\in\omega} u_{x,h_i}\}$.

Theorem 0 (b) then follows from Theorem 5 by the following theorem of van Douwen: if there is a λ-scale for some ordinal λ, then the countable box product of compact metrizable spaces is paracompact.

PROOF OF 0 (b). Assume $\mathbf{c} = \omega_2$. If (*) is false, then there is a dominating family of size ω_1, which is easily seen to contain an ω_1-scale. So we are either in the situation of Theorem 5, or the hypothesis of van Douwen's theorem.

(*) is a statement about ${}^{\omega}\omega$ and the power set of ω; no topological spaces are mentioned. (*) is also implicitly a statement about $\nabla^{\omega}(\omega + 1)$: it says that: (†) there is a basis $\mathfrak{B}$ so that the union of less than $\mathbf{c}$ sets from $\mathfrak{B}$ is closed. The proof of Theorem 5 is also a proof that (†) holds for any $\nabla_{i\in\omega} X_i$, if each X_i is first countable, Lindelöf, and regular.

For our next principle we are not so lucky. We need to know about more than the structure of ${}^{\omega}\omega$ and the power set of ω. We want a method of proof that stratifies a space X under consideration so that we may simultaneously cover each layer. Therefore we do not have a general combinatorial principle, but one which depends on the particular space.

DEFINITION 6. Let $X = \nabla_{i\in\omega} X_i$. $(**)_X$ is the statement that for some ordinal λ there exist $X_\alpha \nearrow X$, $\alpha < \lambda$; $\mathcal{Q}_\alpha \nearrow \mathcal{P}(\omega)$, $\alpha < \lambda$; $\mathcal{C}_\alpha \nearrow \omega\omega$, $\alpha < \lambda$; and $\mathcal{F} = \{f_\alpha : \alpha < \lambda\} \subset {}^\omega\omega$ such that

(1) $\alpha < \beta \Rightarrow f_\beta \in \mathcal{C}_\alpha$,

(2) $f \in \mathcal{C}_\alpha, A \in \mathcal{Q}_\alpha \Rightarrow f \not> f_\alpha$ on A,

(3) $f, g \in \mathcal{C}_\alpha \Rightarrow f + g \in \mathcal{C}_\alpha$,

(4) X_α is Hausdorff under the topology generated by all $u_{x,f}$, where $x \in X_\alpha$, $f \in \mathcal{C}_\alpha$,

(5) if $x, y \in X_\alpha, f, g \in \mathcal{C}_\alpha$, and $u_{x,f} \cap u_{y,g} = \varnothing$, then $\{i: u_{x_i}(f(i)) \cap u_{y_i}(g(i)) = \varnothing\} \in \mathcal{Q}_\alpha$.

Notes on $(**)_X$. 1. Clauses 3, 4, and 5 simultaneously stratify X, ${}^\omega\omega$, and $\mathcal{P}(\omega)$ so that if we only know about the functions and sets on the αth layer, we already know that X_α is Hausdorff. Clauses 1 and 2 connect this stratification with a sequence of functions which may not be dominating, but which no level can dominate.

2. $(**)_X$ is a simplified version of the combinatorics in **[Ro]**.

3. $(*)$ implies $(**)_X$ if $X = \nabla_{i\in\omega} X_i$ and each X_i is first countable, Lindelöf, and regular. Also the existence of a λ-scale for some λ implies $(**)_X$ if $X = \nabla^\omega(\omega + 1)$.

4. As the complication of Definition 6 and the anthropomorphism of Note 1 indicate, $(**)_X$ is designed to tell us when a forcing extension makes box products paracompact.

THEOREM 7. *Let $X = \nabla_{i\in\omega} X_i$, where each X_i is regular and first countable. Then $(**)_X$ implies that X is paracompact.*

PROOF. Again we proceed by induction, this time on the λ of Definition 6. Let $\mathcal{U}$ be an open covering of X. Suppose at stage $\alpha < \lambda$ we have covered a subset of $\bigcup_{\beta<\alpha} X_\beta$ by a disjoint refinement $\mathcal{U}_\alpha$ of $\mathcal{U}$, where each $v \in \mathcal{U}_\alpha$ is clopen and of the form $\bigcap_{n\in\omega} u_{x,nf}$ for some $x \in \bigcup_{\beta<\alpha} X_\beta$, $f \in \bigcup_{\beta<\alpha} \mathcal{C}_\alpha$ (and $nf(j) = n \cdot f(j)$ for all $j \in \omega$). Let $\mathcal{V}$ be the collection of all $u_{y,g}$, for $y \in X_\alpha$, $g \in \mathcal{C}_\alpha$. By $(**)_X$, $\mathcal{V}$ separates points in X. If $y \in X_\alpha$ has not been covered yet, let $h_y > f_\alpha$ where u_{y,h_y} refines a neighborhood in $\mathcal{U}$, $h_y \in \mathcal{C}_{\alpha+1}$, if such an h_y exists. (The problem is that no candidate for h_y may be in $\mathcal{C}_{\alpha+1}$.) If h_y exists, let $u_y = \bigcap_{n\in\omega} u_{y,nh_y}$ and let $\mathcal{U}_{\alpha+1} = \mathcal{U}_\alpha \cup \{u_y : y \in X_\alpha - \bigcup \mathcal{U}_\alpha$ and h_y is defined$\}$. Clauses 2, 4, and 5 of the definition guarantee that $\mathcal{U}_{\alpha+1}$ is a disjoint refinement. Clause 3 says that each nh_y is in $\mathcal{C}_{\alpha+1}$. That every y is eventually covered follows from 1, 3, and the fact that $\mathcal{C}_\alpha \nearrow {}^\omega\omega$.

3. Forcing extensions. In what models of set theory do $(*)$ and $(**)_X$ hold? We partially answer this question with two propositions, the first a simple criterion for $(*)$, and the second a criterion for $(**)_X$. We give examples in each category so the reader will know what we are talking about.

First, some definitions. Let **B** be a complete Boolean algebra. **B** has κ-cc iff every set of mutually incompatible elements has cardinality $< \kappa$. Note that $\kappa < \lambda$ and **B** has κ-cc $\Rightarrow$ **B** has λ-cc. The density of **B**, $d(\mathbf{B})$, is the least cardinality of a dense partial order in **B**.

B has cofinality δ iff there is a sequence of algebras $\mathbf{B}_\alpha \nearrow \mathbf{B}$, $\alpha < \lambda$, where $\mathrm{cf}(\lambda) = \delta$ and each $\mathbf{B}_\alpha$ is a complete proper subalgebra of **B**. Note that the cofinality of **B** is not unique.

Let V be a model of set theory, $\mathbf{B} \in V$, and $\{\mathbf{B}_\alpha: \alpha < \lambda\} \in V$ be a witness that **B** has cofinality δ. We define $V_0 = V$, and $V_\alpha = V^{\mathbf{B}_\alpha} \subset V^{\mathbf{B}}$ for $\alpha > 0$. A function f in $V^{\mathbf{B}}$ is free over V_α iff for every $A, g \in V, f \not< g$ on A. Otherwise f is dominated by V_α. $V^{\mathbf{B}}$ is dominated by V_α iff every f in $V^{\mathbf{B}}$ is dominated by V_α. Otherwise $V^{\mathbf{B}}$ is free over V_α.

A peculiar aspect of forcing is that V often knows the size of the continuum in $V^{\mathbf{B}}$. That is, there is usually a $\kappa \in V$ so that $V \models \kappa = \mathbf{c}^{V^{\mathbf{B}}}$. Call this cardinal $\mathbf{c}^{\mathbf{B}}$.

LEMMA 8. *Suppose $V \models$ [**B** has cofinality $\mathbf{c}^{\mathbf{B}}$ and $V^{\mathbf{B}}$ is free over all V_α]. Then $(*)$ holds in $V^{\mathbf{B}}$.*

PROOF. If $(*)$ fails, there is a dominating family which is contained in some V_α, contradicting the freedom of $V^{\mathbf{B}}$.

EXAMPLES. 1. Any direct iterated ω_1-cc **B** with cofinality $d(\mathbf{B})^\omega$.

2. Mixing things up: e.g. CH is true in each V_α, $\delta = \omega_2$, and each $\mathbf{B}_\alpha$ is a product or iteration of two algebras, one adding at least one free function, the other adding no new reals. Mathias and Laver forcing are examples.

LEMMA 9. *Suppose $V \models$ [**B** has κ^+-cc and cofinality $\delta \geqslant \kappa^+$; $V^{\mathbf{B}}$ is free over all V_α]. Then $(**)_X$ holds in $V^{\mathbf{B}}$ for every $X = \nabla_{i \in \omega} X_i$ where each X_i is regular, first countable with weight $\leqslant \delta$.*

PROOF. The point of the hypothesis on δ is that we can then repeat the construction of **[Ro]** to stratify X into X_α's. A sketch of the construction is:

To each X_i we associate L_i, the lattice of basic open sets. By hypothesis, each L_i is small enough so any countable descending chain is an element of some V_α. We identify a point in X_i with some countable descending chain converging to it. Then $(X_i)_\alpha$ is the collection of such chains which are elements of V_α; $X_\alpha = V_\alpha \cap \nabla_{i \in \omega}(X_i)_\alpha$.

${}^\omega\omega \cap V^{\mathbf{B}}$ and $\mathcal{P}(\omega) \cap V^{\mathbf{B}}$ are already stratified by the V_α's, and this natural stratification, together with the X_α's as above, makes clauses 3, 4, and 5 of $(**)_X$ true. Because $V^{\mathbf{B}}$ is free, we can find an $\mathcal{F}$ satisfying clause 2; we can then skip up the levels to satisfy 1.

EXAMPLES. 1. Any direct ω_1-cc extension with uncountable cofinality.

2. $\mathbf{B} = \mathbf{C} \times \mathbf{D}$, **C** stratifies so that it is free over each $V^{\mathbf{C}_\alpha \times \mathbf{D}} = V_\alpha$, and every real is in some V_α. Candidates for such **C**'s are ω_1-cc algebras, and for the associated **D** good candidates are Solovay, Sacks, or Silver forcing which

all have the property that new functions in ${}^{\omega}\omega$ are dominated by old functions.

3. **B** is an iteration $\mathbf{C} * \mathbf{D}$, and defining $\mathbf{D}_\alpha = \{p: V^{\mathbf{C}_\alpha} \models p \in \mathbf{D}_\alpha\}$, each real occurs in some $V_\alpha = V^{\mathbf{C}_\alpha \times \mathbf{D}_\alpha}$ and $V^{\mathbf{C}}$ is free over each V_α. The same candidates for **C** and **D** in 2 are candidates here, although care must be taken so that $\mathbf{C}_\alpha \times \mathbf{D}_\alpha \nearrow \mathbf{B}$.

Looking at these and other examples, the following questions occur:

C. Must reals be added to destroy paracompact box products? (Yes if, say, (*) or "there is a λ-scale" hold in the ground model. What about other cases?)

D. Can an ω_1-cc extension by itself destroy paracompact box products?

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