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SELF HOMOTOPY EQUIVALENCES OF POSTNIKOV CONJUGATES[1]

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ABSTRACT. The purpose of this note is to generalize a result of Wilkerson [4] and show that the self homotopy equivalences of the Postnikov approximations of a space X determine, in a rather simple manner, simultaneously the self homotopy equivalences of X and the self homotopy equivalences of the Postnikov conjugates of X (i.e. the spaces with the same Postnikov approximations as X).

1. Statement of results. For a connected CW-complex X, let EX denotes its *space of self homotopy equivalences*, B_{EX} the *classifying space* of EX and $X^{(n)}$ the *nth Postnikov approximation* of X. Consider the *homotopy inverse limit space* [**1**, p. 301]

$$V = \text{ho} \underset{\leftarrow}{\text{proj}} \lim \ B_{EX^{(n)}}$$

i.e. the space obtained by turning the tower $\{B_{EX^{(n)}}\}$ into a tower of fibrations and taking the ordinary limit of the latter. Then it is reasonable to expect that this space V is in some way related to B_{EX} and it turns out that, in fact, *one component of* V, the one "containing" the identity maps of the $X^{(n)}$, *has exactly the homotopy type of* B_{EX}. To describe the homotopy types of the (possible) other components of V one needs the *Postnikov conjugates* of X, i.e. the spaces Y for which each $Y^{(n)}$ has the same homotopy type as $X^{(n)}$ (such spaces need *not* have the same homotopy type as X, as the homotopy equivalences $Y^{(n)} \to X^{(n)}$ are *not* required to be compatible). Our main result then is:

1.1 THEOREM. (i) *There is a canonical* 1-1 *correspondence between the components of* V *and the homotopy types of the Postnikov conjugates of* X.

(ii) *For every Postnikov conjugate* Y *of* X, *the corresponding* (*see* (i)) *component* $V_Y \subset V$ *has the homotopy type of* B_{EY}.

As the homotopy groups of the homotopy inverse limit of a tower fit into short exact sequences involving a proj lim and a proj lim^1 term [**1**, p. 310], Theorem 1.1 implies the following "known" result:

1.2 COROLLARY. *For every Postnikov conjugate* Y *of* X *and every integer* $i \geqslant 0$, *there is a short exact sequence*

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$$* \to \text{proj lim}^1 \pi_{i+1} EX^{(n)} \to \pi_i EY \to \text{proj lim}\, \pi_i EX^{(n)} \to *$$

in which each of the maps $\pi_j EX^{(k)} \to \pi_j EX^{(k-1)}$ *involved in the* proj lim *and* proj $\lim^1$ *terms differs from the obvious one by the action of an element of* $\pi_0 EX^{(k-1)}$ (*which depends on* Y).

1.3 COROLLARY. *There is a canonical* 1-1 *correspondence between the elements of the set* proj $\lim^1$ $\pi_0 E_X^{(n)}$ *and the homotopy types of Postnikov conjugates of* X.

As $\pi_0 EX^{(n)}$ is the *group of homotopy classes of self homotopy equivalences of* $X^{(n)}$, this corollary is just Theorem I of Wilkerson [4]. In fact, it was our attempt to understand this result of Wilkerson which led to our Theorem 1.1.

1.4 *Generalizations.* If X has a *base point*, then one can consider the space $E_* X$ of *base point preserving self homotopy equivalences* and it is not hard to verify that Theorem 1.1 and its proof remain valid if one replaces everywhere E by E_*. One could go even further and consider, for instance, the space $E_\pi X$ of *base point preserving self homotopy equivalences which induce the identity on all the homotopy groups*. Again Theorem 1.1 and its proof remain valid if one replaces everywhere E by E_π, except that one has to suitably restrict the notion of Postnikov conjugate.

1.5 *Organization of the proof.* In order to keep the rather technical proof of our theorem as reasonable as possible, we reduce (in §2) Theorem 1.1 to a *minimal simplicial version* 2.1 and then proceed (in §3) to prove the latter. This has the considerable advantage that one then can use *isomorphisms* instead of *homotopy equivalences* as, for minimal simplicial sets, these two notions coincide. We will freely use the simplicial terminology and notation of [1] and [2].

2. Reduction to the minimal simplicial case. Let M be a connected simplicial set which is *minimal* [2, p. 35] and denote by $M^{(n)}$ its *nth Postnikov approximation* [2, p. 34]. By a *Postnikov conjugate* of M we then mean any minimal simplicial set N for which each $N^{(n)}$ is isomorphic to $M^{(n)}$ (again this does *not* imply that N is isomorphic to M, as the isomorphisms $N^{(n)} \approx M^{(n)}$ are *not* required to be compatible). If AM denotes the *simplicial group of automorphisms* of M [2, p. 74] and $\overline{W}$ is the *simplicial classifying functor* [2, p. 87], then one can form the *homotopy inverse limit* [1, p. 295]

$$L = \text{ho proj lim}\, \overline{W} AM^{(n)}$$

and formulate the following minimal simplicial version of Theorem 1.1.

2.1 THEOREM. (i) *There is a canonical* 1-1 *correspondence between the components of L and the* (*isomorphism classes of*) *Postnikov conjugates of M*.

(ii) *For every Postnikov conjugate N of M, the corresponding* (*see* (i)) *component* $L_N \subset L$ *has the homotopy type of* $\overline{W} AN$.

THEOREM 1.1 now readily follows from this minimal simplicial version by applying Theorem 2.1 to a *minimal subcomplex of the singular complex* of X and observing that, *for a minimal simplicial set* M, *every self homotopy equivalence is an automorphism and, in fact*, $EM = AM$.

It thus remains to prove Theorem 2.1, and this we will do in §3.

3. Proof of Theorem 2.1. The main part of the proof consists of, given a Postnikov conjugate N of M and a choice of isomorphisms $q\colon N^{(n)} \approx M^{(n)}$, constructing an explicit map $\overline{W}AN \to L$. Once this is done, it will not be hard to show that

3.1(i) *The correspondence* $N \to L_N$, *which assigns to* N *the component of* L *containing the image of* $\overline{W}AN$ *under this map, does not depend on the choice of the* q*'s and, in fact, induces a* 1-1 *correspondence between the* (*isomorphism classes of*) *Postnikov conjugates of* M *and the components of* L, *and*

3.1(ii) *The map* $\overline{W}AN \to L_N$ *is a homotopy equivalence.*

We start with constructing a commutative diagram

$$\begin{array}{ccccccc} \cdots & \to & \overline{W}AN^{(n+1)} \times I/n+1 & \to & \overline{W}AN^{(n)} \times I/n & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \overline{W}AM^{(n+1)} & \to & \overline{W}AN^{(n)} & \to & \cdots \end{array} \tag{3.2}$$

in which I/n denotes the (contractible) simplicial set that has [**1**, p. 292] a k-simplex for every nondecreasing sequence of integers $(n, n_0, \ldots, n_k)$. (The existence of this commutative diagram is equivalent to the statement that the diagram

$$\begin{array}{ccccccc} \cdots & \to & \overline{W}AN^{(n+1)} & \to & \overline{W}AN^{(n)} & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \overline{W}AM^{(n+1)} & \to & \overline{W}AM^{(n)} & \to & \cdots \end{array}$$

in which the vertical maps are induced by the q's, *commutes up to compatible homotopies* [3]). The horizontal maps in (3.2) are the obvious ones and the vertical map $\overline{W}AN^{(n)} \times I/N \to \overline{W}AM^{(n)}$ is given by the formula

$$((b_{k-1}, \ldots, b_0), (n, n_0, \ldots, n_k)) \to \left(a_{k-1}qb_{k-1}q^{-1}a_k^{-1}, \ldots, a_0qb_0q^{-1}a_1^{-1}\right)$$

in which $a_{k-i}\colon M^{(n)} \approx M^{(n)}$ denotes the *unique* isomorphism such that the diagram

$$\begin{tikzcd} & N^{(n_i)} \arrow[r, "q"] \arrow[dl, "\mathrm{proj}"'] & M^{(n_i)} \arrow[dr, "\mathrm{proj}"] & \\ N^{(n)} \arrow[rr, "q"] & & M^{(n)} \arrow[r, "a_{k-i}"] & M^{(n)} \end{tikzcd}$$

commutes and in which q, q^{-1}, a_i and a_i^{-1} are abbreviations for $q \times \Delta[j]$, $q^{-1} \times \Delta[j]$, $a_i \times \Delta[j]$ and $a_i^{-1} \times \Delta[j]$ with appropriate j. A straightforward but long calculation then shows that the function $\overline{W}AN^{(n)} \times I/n \to \overline{W}AM^{(n)}$

so defined is actually a simplicial map and that diagram (3.2) indeed commutes.

Next we observe that another long but straightforward calculation yields

3.3. *The diagram* (3.2) *induces an isomorphism*

$$\text{ho proj lim } \overline{W}AN^{(n)} \overset{r}{\approx} \text{ ho proj lim } \overline{W}AM^{(n)} = L$$

by assigning to every k-simplex of ho proj lim $\overline{W}AN^{(n)}$, *i.e. compatible collection of maps* $\Delta[k] \times I/n \to \overline{W}AN^{(n)}$, *the composition*

$$\Delta[k] \times I/n \to \Delta[k] \times I/n \times I/n \to \overline{W}AN^{(n)} \times I/n \to \overline{W}AM^{(n)}.$$

Finally we obtain the desired map $\overline{W}AN \to L$ as the obvious [**1**, p. 297] composition

$$\begin{aligned}\overline{W}AN &= \text{proj lim } \overline{W}AN^{(n)} \to \text{ho proj lim } \overline{W}AN^{(n)} \\ &\overset{r}{\approx} \text{ho proj lim } \overline{W}AM^{(n)} = L.\end{aligned}$$

One now proves 3.1(i) by verifying what happens to the vertex of $\overline{W}AN$ under the map $\overline{W}AN \to L$. The argument is straightforward and is essentially a $\overline{W}$-translation of the proof of Theorem I of Wilkerson [**4**].

To prove 3.1(ii) one notes that the projections $N^{(n+1)} \to N^{(n)}$ are fibrations and that therefore the induced maps $AN^{(n+1)} \to AN^{(n)}$ are also fibrations (even though they need not be onto). It readily follows [**1**, p. 254 and p. 310] that the induced [**1**, p. 297] map

$$\pi_i \overline{W}AN = \pi_i \text{ proj lim } \overline{W}AN^{(n)} \to \pi_i \text{ ho proj lim } \overline{W}AN^{(n)}$$

is an isomorphism for all $i \geqslant 1$ and the desired result thus follows from 3.3.

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