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ON HOMOMORPHISMS FROM ALGEBRAIC GROUPS TO $\mathrm{GL}_n(\mathbf{Z})$

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ABSTRACT. As a rule, there are no nontrivial abstract group homomorphisms from a connected algebraic group to any group devoid of divisible elements. This rule is formulated and proven, and the exceptions to the rule as well as applications of it are discussed.

In this note we consider the following question: Are there nontrivial abstract group homomorphisms from an algebraic group to the integral modular group $\mathrm{GL}_n(\mathbf{Z})$ of rank n? The answer, as one would expect, is no as a rule, but significant exceptions do arise. First of these exceptional cases is the one in which the ground field has a positive characteristic and the group is not reductive. The second exceptional case occurs when the ground field is not algebraically closed, even when it is such a commonplace field as the real numbers.

By an algebraic group we mean linear (i.e., affine) group scheme, algebraic and smooth over a field k. We assume k to be algebraically closed and identify all algebraic groups with their respective k-rational point sets.

Actually, our results trivially generalize to group varieties (abelian varieties and the like). Also, they have some minor applications in algebraic geometry. Those are described below.

Elementary known facts from algebraic group theory have been liberally used in what follows. They are all found in Borel's book [1].

Our *notations* are standard: $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$, respectively, stand for the integers, the reals and the complex number field. $\mathbf{G}_a$ and $\mathbf{G}_m$ denote respectively the one-dimensional vector group and the one-dimensional torus both defined over k.

By definition, an element X of a group is said to be *divisible* if for each positive integer r there is a group element Y such that $X = Y^r$. As usual, a group is called divisible if its every element is divisible.

LEMMA. *No element of* $\mathrm{GL}_n(\mathbf{Z})$ *other than the identity element* I *is divisible.*

PROOF. Assume the contrary and let $X \neq I$ be divisible. Choose a prime p such that $X \not\equiv I \pmod{p}$, and call r the order of the finite group $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$. By assumption, there is an element $Y \in \mathrm{GL}_n(\mathbf{Z})$ such that $X = Y^r$. Then,

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reducing everything *modulo p*, one gets $X \equiv Y^r \equiv I \pmod{p}$, a contradiction. Q.E.D.

THEOREM. *A connected, reductive algebraic group over an algebraically closed field has no nontrivial homomorphisms into* $\mathrm{GL}_n(\mathbf{Z})$.

PROOF. As any connected reductive algebraic group is a union of its Borel subgroups, it suffices to prove our assertion for each of the Borel subgroups. Let B be one such group. Then $B = U \rtimes T$, a semidirect product of its unipotent radical U and a maximal torus T. Further, there exists an ordering on the root system relative to T such that if Φ^+ is the set of positive roots for this ordering then to each $\alpha \in \Phi^+$ corresponds a unique subgroup $U_\alpha \subseteq U$ with an isomorphism $j_\alpha\colon U_\alpha \xrightarrow{\sim} \mathbf{G}_a$ such that

(i) the U_α's generate U, and

(ii) $txt^{-1} = j_\alpha^{-1}(\alpha(t)j_\alpha(x))$ for all $t \in T$, all $x \in U_\alpha$.

Now let $h\colon B \to \mathrm{GL}_n(\mathbf{Z})$ be a homomorphism. Since every $t \in T$ is clearly divisible, so must be $h(t)$. Then, $h(t) = I$ (the identity) by Lemma, and h is trivial on T. Therefore, in view of (i) just above, we need only show $h(x) = I$ for all $x \in U_\alpha$ in order to establish $h(B) = \{I\}$. But, by (ii) above, we have

$$\begin{aligned} x^{-1}txt^{-1} &= x^{-1}j_\alpha^{-1}(\alpha(t)j_\alpha(x)) \\ &= j_\alpha^{-1}(-j_\alpha(x))j_\alpha^{-1}(\alpha(t)j_\alpha(x)) \\ &= j_\alpha^{-1}\big((\alpha(t) - 1)j_\alpha(x)\big) \end{aligned}$$

for any $t \in T$. Choose a value t_1 of t so that $\alpha(t_1) - 1 = \alpha(s)$ for some $s \in T$. Then,

$$x^{-1}t_1xt_1^{-1} = j_\alpha^{-1}(\alpha(s)j_\alpha(x)) = sxs^{-1}.$$

Apply h to both ends of this equality and we get $h(x) = I$. Q.E.D.

COROLLARY. *A connected algebraic group over an algebraically closed field of characteristic zero has no nontrivial homomorphisms into* $\mathrm{GL}_n(\mathbf{Z})$.

PROOF. Let G be such an algebraic group, and let U be its unipotent radical. Since U is a multiple extension of $\mathbf{G}_a$'s and $\mathbf{G}_a$ is divisible, any homomorphism $G \to \mathrm{GL}_n(\mathbf{Z})$ must be constant on U by Lemma. So, a homomorphism $G/U \to \mathrm{GL}_n(\mathbf{Z})$ is induced, which is constant by Theorem. Q.E.D.

REMARK 1. The theorem above without the reductivity assumption is false in positive characteristics. For example, take $\mathbf{G}_a = (k, +)$ over an algebraically closed field k of characteristic $p > 0$. Consider $\mathbf{G}_a$ as a vector space over the prime field $GF(p) \subset k$, fix a basis, and take any projection map

$$\mathbf{G}_a \to GF(p) \simeq \mathbf{Z}/p\mathbf{Z}$$

onto a coordinate axis. This is a nontrivial homomorphism. Now it is easy to embed $\mathbf{Z}/p\mathbf{Z}$ into $\mathrm{GL}_n(\mathbf{Z})$ as a subgroup, for many values of n. Also from this argument, one can see at once that in both Theorem and Corollary the connectivity assumption is of the essence.

REMARK 2. The following two examples show that this type of result does not apply to the group of k-rational points of a connected algebraic group if k is not algebraically closed. First, the group of real points of $\mathbf{G}_m$ is just $\mathbf{R}^\times$, and this has the subgroup $\mathbf{R}^+$ of index 2, and thus a nontrivial homomorphism into any $\mathrm{GL}_n(\mathbf{Z})$. For an example in which G is simple, let $G = \mathrm{PGL}_2$. Then $\mathrm{PGL}_2(\mathbf{R})$ is not simple, having the normal subgroup $\mathrm{SL}_2(\mathbf{R})/\{\pm 1\}$ of index 2, thus a nontrivial homomorphism into any $\mathrm{GL}_n(\mathbf{Z})$.

Continuing to work over an algebraically closed ground field k, we now consider more generally a *k-group variety* G. By this is meant an irreducible algebraic k-group scheme smooth over k, and as before we shall identify such G with its k-rational points. According to a well-known theorem of Chevalley, there is a unique maximal linear connected normal k-closed subgroup L such that G/L is an abelian variety. A k-group variety G is called *reductive* if L is reductive, and G is called a *quasi-abelian variety* if L is an algebraic torus. The last condition is easily seen to be equivalent to G's containing no subgroup isomorphic to $\mathbf{G}_a$.

We now draw some easy consequences of the definitions and the results proven above. Let us begin by restating the main results in utmost generality:

(a) *Let G be a k-group variety, and assume G to be reductive if* $\mathrm{char}(k) \neq 0$. *Then, there are no nontrivial homomorphisms from G to any group devoid of divisible elements $\neq$ the identity.*

This is clear if one recalls that the underlying group of an abelian variety is always divisible and that the absence of divisible elements was all that one needed from $\mathrm{GL}(n, \mathbf{Z})$ in the foregoing proofs.

(b) *A quasi-abelian variety is commutative.* (Compare Rosenlicht [**3**], Iitaka [**2**, Lemma 4, p. 186].)

In fact, given an exact sequence $1 \to T \to G \to A \to 1$ with T a torus and A an abelian variety, the mapping $g \in G \mapsto (\phi_g\colon T \to T)$ defined by $\phi_g(t) = gtg^{-1}$ for all $t \in T$ is a homomorphism from G into the automorphism group of T. Since this last is isomorphic to a $\mathrm{GL}_n(\mathbf{Z})$, ϕ_g is the identity and T is therefore central in G. Then the commutator function $G \times G \to T$ must factor through $G \times G \to A \times A$ *via* a morphism $A \times A \to T$. Since $A \times A$ is complete over k, this morphism is constant. So, the commutator is trivial.

(c) Let G be a k-group variety, reductive if $\mathrm{char}(k) \neq 0$, and suppose that G operates on an affine normal variety X. For any rational function f on X and for any $s \in G$, define the s-transform f^s of f by the rule $f^s(x) = f(s \cdot x)$ for all $x \in X$ provided f is regular at $s \cdot x$.

Let f be an invertible regular function on X. Then, there exists a rational character $\chi\colon G \to \mathbf{G}_m$ (i.e., a morphism of group varieties) such that $f^s = \chi(s) f$ for all $s \in G$. (Compare Sumihiro [**4**, Lemma 2, p. 3].)

To prove this, embed X as an open subvariety of a projective normal variety Y. The boundary $Y - X$ is then easily seen to be pure of codimension one, so write $Y - X = D_1 \cup \cdots \cup D_n$ with D_i's distinct hypersurfaces of Y. The correspondence (invertible regular function f on X)$\mapsto$(divisor of zeros and poles of f on Y) gives an isomorphism of abelian groups {invertible

regular functions under multiplication}/{nonzero constants} and the free $\mathbf{Z}$-module generated by $D_1, \ldots, D_n$. Since G acts naturally on the former group, it does on the latter which is isomorphic to $\mathbf{Z}^n$. Thus a homomorphism $G \to \mathrm{GL}_n(\mathbf{Z})$ obtains, which is reduced to a constant by the results above. Therefore, for any invertible regular function f and any $s \in G$, $f^s = \chi(s)f$ with $\chi(s) \in k$. Clearly, χ is a rational character.

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