ON THE PROBLEM OF PARTITIONING
\{1, 2, \ldots, n\} INTO SUBSETS HAVING EQUAL SUMS

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ABSTRACT. Let \( N \) denote the set of natural numbers and let \( Z_n = \{1, 2, \ldots, n\} \). For \( S \) a finite subset of \( N \), let \( \sigma S \) denote the sum of the elements in \( S \). Then \( \sigma Z_n = n(n + 1)/2 \). Suppose \( n(n + 1) = 2st \), where \( s \) and \( t \) are integers and \( t > n \). We show that \( Z_n \) can be partitioned into \( T_1 \cup T_2 \cup \ldots \cup T_s \) such that \( \sigma T_i = t \), for \( 1 < i < s \). Such a partition is called an \((s, t)\)-partition of \( Z_n \).

A graph \( G \) having \( n(n + 1)/2 \) edges is said to be path-perfect if the edge set of \( G \) can be partitioned as \( E_1 \cup E_2 \cup \ldots \cup E_s \) so that \( E_i \) induces a path of length \( i \), for \( 1 < i < n \). Suppose \( p \) and \( n \) are positive integers and \( r \) is an even positive integer with \( p > r + 1 \) and \( pr = n(n + 1) \). The existence of an \((r/2, p)\)-partition of \( Z_n \) is used to show the existence of an \( r \)-regular path-perfect graph \( G \) having \( p \) vertices and \( n(n + 1)/2 \) edges.

1. Introduction. Let \( N \) denote the set of natural numbers and let \( Z_n = \{1, 2, \ldots, n\} \). For \( S \) a finite subset of \( N \), let \( \sigma S \) denote the sum of the elements belonging to \( S \).

It is well known that \( \sigma Z_n = n(n + 1)/2 \). We consider the following question. Suppose \( n(n + 1) = 2st \), where \( s \) and \( t \) are integers and \( t > n \). Do there exist \( s \) pairwise disjoint subsets of \( Z_n \), say \( T_1, T_2, \ldots, T_s \), such that \( Z_n = T_1 \cup T_2 \cup \ldots \cup T_s \) and \( \sigma T_i = t \), for each \( i, 1 < i < s \)? If so, then the partition \( T_1 \cup T_2 \cup \ldots \cup T_s \) will be called an \((s, t)\)-partition of \( Z_n \). As an example, let \( n = 20, s = 6, \) and \( t = 35 \). The subsets \( T_1 = \{15, 20\}, T_2 = \{16, 19\}, T_3 = \{17, 18\}, T_4 = \{4, 8, 9, 14\}, T_5 = \{5, 7, 10, 13\}, \) and \( T_6 = \{1, 2, 3, 6, 11, 12\} \) form a \((6, 35)\)-partition of \( Z_{20} \).

This number theoretical problem arose in the context of a problem in graph theory. In [3], Straight considered the question of the existence of regular path-perfect graphs, a concept to be explained in §3. While doing so, he was able to find a partial solution to the problem concerning \((s, t)\)-partitions of \( Z_n \). He posed the problem during a talk at SUNY-Fredonia, and from there it was communicated to Schillo, who found a complete solution.

In §2 we show that an \((s, t)\)-partition of \( Z_n \) exists whenever \( n(n + 1) = 2st \). In §3 this result is then applied to the question of the existence of even-regular, path-perfect graphs.

2. The main result.

THEOREM 1. Let \( n \) be a positive integer and suppose \( n(n + 1) = 2st \), where \( s \) and \( t \) are integers and \( t > n \). Then there exists an \((s, t)\)-partition of the set \( Z_n \).

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Proof. We employ induction on \( n \). It is easy to verify the theorem for small values of \( n \). Now let \( n \) be some fixed positive integer and suppose the theorem is true whenever \( \hat{n}(\hat{n} + 1) = 2\hat{s}\hat{t}, \hat{t} \geq \hat{n}, \) and \( \hat{n} < n \). Let \( s \) and \( t \) be given with \( n(n + 1) = 2st \) and \( t > n \). We wish to find sets \( T_1, T_2, \ldots, T_s \) forming an \((s, t)\)-partition of \( \mathbb{Z}_n^n \).

**Case 1A.** Suppose \( n < t < 2n \) and \( t \) is even. Let \( m = (2n - t)/2 \) and for \( 1 \leq i < m \), let \( T_i = \{n - i + 1, t - n + i - 1\} \). Note that the elements of \( \mathbb{Z}_n^n \) not belonging to \( T_1 \cup T_2 \cup \ldots \cup T_m \) form the set \( \mathbb{Z}_{t-n-1} \cup \{t/2\} \). Now \( \sigma_{\mathbb{Z}_{t-n-1}} = (t/2)(t - 2n + 2s - 1) \) and \( t - n - 1 < n \). Thus, by the induction hypothesis, there exists a \((t - 2n + 2s - 1, t/2)\)-partition of \( \mathbb{Z}_{t-n-1} \). Combine one of the sets in this partition with \( \{t/2\} \) to obtain \( T_{m+1} \), and unite the rest two at a time to form \( T_{m+2}, \ldots, T_s \). Then \( T_1 \cup T_2 \cup \ldots \cup T_s \) is an \((s, t)\)-partition of \( \mathbb{Z}_n^n \).

**Case 1B.** Suppose \( n < t < 2n \) and \( t \) is odd. Let \( k = (2n - t + 1)/2 \) and for \( 1 \leq i < k \), let \( T_i = \{n - i + 1, t - n + i - 1\} \). After forming \( T_1, \ldots, T_k \), the subset of \( \mathbb{Z}_n^n \) which remains is \( \mathbb{Z}_{t-n-1} \). Now \( \sigma_{\mathbb{Z}_{t-n-1}} = ((t - 1 - 2n + 2s)/2, t) \)-partition of \( \mathbb{Z}_{t-n-1} \). Letting the elements of this partition be \( T_{k+1}, \ldots, T_s \), we obtain an \((s, t)\)-partition of \( \mathbb{Z}_n^n \).

**Case 2.** Suppose \( t > 2n \). Using the induction hypothesis, let \( T_1' \cup T_2' \cup \ldots \cup T_s' \) be an \((s, t - 2n + 2s - 1)\)-partition of \( \mathbb{Z}_{n-2s} \). Next let \( T_i = T_i' \cup \{n - 2s + i, n - i + 1\} \) for \( 1 \leq i < s \). Then \( T_1 \cup T_2 \cup \ldots \cup T_s \) is an \((s, t)\)-partition of \( \mathbb{Z}_n^n \).

Therefore, by induction, the theorem is proven. \( \square \)

3. **An application to graph theory.** Let \( K_n \) and \( P_n \) denote the complete graph of order \( n \) and the path of length \( n \), respectively. A factorization of a graph \( G \) is a partition of its edge set. The subgraphs of \( G \) induced by the elements in the partition are called factors. In [1], Fink and Straight consider the problem of factoring a graph into paths of different lengths. Specifically, they define a graph \( G \) having size \( n(n + 1)/2 \) to be path-perfect if the edge set of \( G \) can be partitioned as \( E_1 \cup E_2 \cup \ldots \cup E_n \), so that \( E_i \) induces \( P_i \), for \( 1 < i < n \). For example, the Petersen graph, shown in Figure 1, is path-perfect. To see this, let \( P_1 = v_7v_8, P_2 = v_4v_9v_{10}, P_3 = v_3v_1v_8v_9, P_4 = v_3v_7v_8v_{10}v_2 \) and \( P_5 = v_1v_2v_3v_4v_5v_6 \).

Note also that the Petersen graph is 3-regular (each vertex is incident with 3 edges). If \( G \) is an \( r \)-regular graph of order \( p \) and size \( n(n + 1)/2 \), then

\[ pr = n(n + 1). \]

Thus \( r \) must divide \( n(n + 1) \). Considering the case where \( r \) divides \( n \) or \( n + 1 \), Straight [3] showed the following.

**Theorem A.** Let \( r \) be an odd positive integer. There exists an \( r \)-regular path-perfect graph of order \( m(mr + 1) \) if, and only if, \( m \) equals 1. Also, there exists an \( r \)-regular path-perfect graph of order \( m(mr - 1) \) if, and only if, \( m \) equals 2.
If \( r \) is even, then equation (1) can be written as \( p(r/2) = n(n + 1)/2 \). We now wish to apply Theorem 1 with \( t = p \) and \( s = r/2 \) to help prove the following.

**Theorem 2.** Let \( p \) and \( n \) be positive integers and let \( r \) be an even positive integer such that \( p > r + 1 \) and \( pr = n(n + 1) \). Then there exists an \( r \)-regular path-perfect graph \( G \) of order \( p \) and size \( n(n + 1)/2 \).

**Proof.** Let \( p, n \) and \( r \) be given satisfying the conditions of the theorem. We shall construct \( G \).

It is a well-known result in graph theory (see [2, p. 89]) that if \( p \) is even, \( K_p \) can be factored into \((p - 2)/2\) hamiltonian cycles and a factor which is 1-regular, while if \( p \) is odd, \( K_p \) can be factored into \((p - 1)/2\) hamiltonian cycles. In either case, we may combine \( r/2 \) of these cycles, call them \( C_1, C_2, \ldots, C_{r/2} \), to form an \( r \)-regular spanning subgraph of \( K_n \). This subgraph will be \( G \).

To show that \( G \) is path-perfect, apply Theorem 1 with \( s = r/2 \) and \( t = p \). For \( 1 < i < s \), the subset \( T_i \) of \( Z_n \) given by the theorem tells us how to partition \( C_i \) into paths; that is, the elements of \( T_i \) are the lengths of the paths.

Theorem 2 answers the question of the existence of \( r \)-regular path-perfect graphs when \( r \) is even. Theorem A can be applied when \( r \) is odd and \( r \) divides \( n \) or \( r \) divides \( n + 1 \). In particular, one can apply Theorem A whenever \( r \) is a power of some odd prime. This leaves one unsolved case—when \( r \) is odd and \( r \) divides neither \( n \) nor \( n + 1 \). For example, we do not know whether there exists a 15-regular path-perfect graph of order 28 and size \( 20(21)/2 = 210 \).

**References**