INFINITE DIMENSIONAL L-SPACES
DO NOT HAVE PREDUALS OF ALL ORDERS

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Abstract. It is shown that if $E$ is an infinite dimensional Banach space with first dual $E'$, second dual $E''$, and $n$th dual $E^{[n]}$ and if $E^{[m]}$ is either an $L$- or $M$-space all duals are either $L$- or $M$-spaces except possibly $E$ which could be a Lindenstrauss space. If $E$ is an $L$- or $M$-space there is an integer $n(E)$ so that if $m > n(E)$ there is no Banach space $F$ with $E = F^{[m]}$. The linear isomorphic analogues to these isometric results are also established. In particular if $E$ is an $L_1$ or $L_\infty$ space there is an integer $n(E)$ so that $E$ is not linearly isomorphic to $F^{[m]}$ for any Banach space $F$ when $m > n(E)$.

Introduction. Consider a sequence $(E_j: j = 1, \ldots, n)$ of Banach spaces with $E_{j+1}' = E_j$ for $j = 1, \ldots, n-1$. We call $E_2$ a predual of $E_1$ and $E_2$ a pre-bidual of $E_1$. In general we say that $E_{j+1}$ is a predual of $E_j$ of order $j$ and denote this fact by the equation $E_j^{[j]} = E_1$. Is it possible that there exists, for a given Banach space $E$, an infinite sequence $(E_j: j \in \mathbb{N})$ of Banach spaces with $E_1 = E$ and with $E_j$ the dual of $E_{j+1}$ for all $j \in \mathbb{N}$? If $E$ is reflexive, in particular if $E$ is finite dimensional, this is easily seen to be true taking $E_j = E'$ if $j$ is even and $E_j = E$ if $E$ is odd. James, [6], constructs an example of a nonreflexive separable Banach space $E$ isometric with its bidual. For such a space $E$ the desired sequence $(E_j: j \in \mathbb{N})$ is readily produced. If $E$ is required to be an infinite dimensional $L$-space the existence of such a sequence is impossible. This follows from the main result of this paper which asserts the existence of an integer, $n(E)$, for any infinite dimensional $L$-space $E$, such that $E$ has no predual of order $n(E)$. If $E$ is an $L$-space it is known that $E^{[2n]}$ is also an $L$-space for all $n$. Thus, for all $n$ there are $L$-spaces $E$ with $n(E)$ arbitrarily large. That $n(E)$ may be 0 was established by Gelfand, [4], for $E = L^1(\mu)$ where $\mu$ is Lebesgue measure on $[0, 1]$ or on $(-\infty, \infty)$, and by Pelczyński, [9], for $\mu$ $\sigma$-finite but not purely atomic. Rosenthal, [13], establishes that when $\Gamma$ is a set whose cardinality, $\text{card}(\Gamma)$, satisfies $\aleph_0 < \text{card}(\Gamma) < 2^\varepsilon$ then $l^\infty(\Gamma)$ does not have a predual of order 3 so that $l^1(\Gamma)$, although a dual space, is not a bidual space. Hence, $n(l^1(\Gamma)) = 1$. We shall remove the cardinality restriction $\Gamma$ in Proposition 2. Thus, if $\mu$ is a $\sigma$-finite measure and $E = L^1(\mu)$ then $n(E) = 0$ unless $\mu$ is purely atomic in which case $n(E) = 1$.

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This appears to be the extent of precise knowledge of \( n(E) \) although we shall give a method for obtaining crude estimates of \( n(E) \) for more general \( L \)-spaces.

1. The preduals of order 2 of an \( L \)-space. In order to proceed we need to characterize Banach spaces whose \( n \)th duals are \( L \)-spaces. It is well known that the dual of an \( L \)-space (\( M \)-space) is an \( M \)-space with unit \( (L \)-space).

If \( F \) is an \( M \)-space which has a predual \( E \) then \( E \) considered as a subspace of \( F' \) is unique and is an \( L \)-space with the induced ordering from the \( L \)-space \( F' \). If \( E \) is an \( L \)-space with predual \( F \) then \( F \) need not be an \( M \)-space with any ordering. Preduals of \( L \)-spaces \( E \) are called Lindenstrauss spaces. If a Lindenstrauss space, when given the induced ordering from \( E' \), is an ordered Banach space with \( E \) as order-dual it is called a simplex space. The classification of Lindenstrauss spaces is an unwieldy task which has absorbed the research of many persons. The situation for preduals of order \( n > 1 \) could be expected to be even more complicated. Somewhat surprisingly the situation is simplified enormously.

**Proposition 1.** Let \( E \) be a Banach space with \( E'' \) an \( L \)-space. \( E' \) is an \( M \)-space and \( E \) is an \( L \)-space.

**Proof.** Let \( B' \) and \( B'' \) be the unit balls of \( E' \) and \( E'' \). There is an extreme point \( e \) of \( B' \) since \( B' \) is \( \sigma(E', E) \) compact. Lindenstrauss, [7], shows that there is a compact Hausdorff space \( X \) and an isometry of \( E' \) onto a separating subspace \( F \) of \( C(X) \) such that \( e \) is mapped to 1 and such that \( F \) has the Riesz separation property with respect to the ordering of \( C(X) \). With the ordering and norm from \( C(X) \) \( F \) is order unit normed with order unit \( e \). Any such space is isometric and order isomorphic with the space \( A(\Delta) \) of continuous affine functions on a Choquet simplex \( \Delta \). Thus \( E' \) is isometric with \( A(\Delta) \). Capon shows, [2], that when \( A(\Delta) \) is a dual space then \( \Delta \) is a Bauer simplex. Thus, \( A(\Delta) \) is an \( M \)-space as is \( E' \). Since \( E \) is the predual of \( E' \) it is an \( L \)-space. □

**Corollary 1-1.** Let \( E \) be a Banach space with no predual. If, for some integer \( n \), \( E^{[n]} \) is an \( L \)- or an \( M \)-space then \( E \) is either an \( L \)-space or a Lindenstrauss space.

**Remarks.** 1. The following simple category theoretic proof of Proposition 1 is due to K. S. Lau. For terminology and justification of all statements we refer the reader to [14]. We work in the category \( \text{Ban}_1 \) of Banach spaces with linear contractions as morphisms. Any dual \( M \)-space is injective and all injective objects in this category are \( M \)-spaces. If \( E'' \) is an \( L \)-space \( E' \) is a retract of the injective \( E''' \) hence is itself injective. Thus \( E' \) is an \( M \)-space and \( E \) is an \( L \)-space.

2. One might conjecture that if \( E \) were a Banach space with \( E'' \) linearly isomorphic to an \( L \)-space then \( E \) would also be linearly isomorphic to an \( L \)-space. A counterexample appears in Chapter 5d of [9]. An example is given
of a Banach space $E$ not even linearly isomorphic to a complemented subspace of an $L$-space yet with $E'$ linearly isomorphic with $l^\infty(N)$. Severe restrictions on $E$ are necessary in order that $E''$ linearly isomorphic to an $L$-space imply the existence of an $L$-space linearly isomorphic with $E$.

**Proposition 2.** If $\Gamma$ is an infinite set then $n(l^1(\Gamma)) = 1$. In fact $l^1(\Gamma)$ is not even linearly isomorphic to a double conjugate space.

**Proof.** Let Ban be the category of Banach spaces with continuous linear maps as morphisms. Let $E$ be a Banach space with $E''$ isomorphic to $l^1(\Gamma)$. $E'''$ is isomorphic to $l^\infty(\Gamma)$ which is injective in Ban$_1$ hence $E'''$ is injective in Ban. Consequently, $E'$ is injective in Ban. Corollary 3 of [12], shows that $E'$ contains a closed subspace $l$ linearly isomorphic to $l^\infty(N)$. Since $l^\infty(N)$ is injective in Ban, $l$ is complemented in $E'$ hence $E''$ contains a complemented subspace linearly isomorphic to $l^\infty(N)$ hence to $l^\infty*(\beta N)$. Since $\beta N$ is not scattered there is a nonatomic measure $\mu \in \mathcal{M}^+(\beta N)$, [13]. If $\mathcal{B}$ is a separable $\mu$-nonatomic $\sigma$-algebra of subsets of $N$ the space $L^1(\beta N, \mathcal{B}, \mu)$ is isometric both with $L^1(\lambda_1)$ where $\lambda_1$ is Lebesgue measure on $[0, 1]$ and with the closed subspace \(|f \cdot \mu: f \in L^1(\beta N, \mathcal{B}, \mu)\) \(\subset \mathcal{M}(\beta N)\). Thus, $l^1(\Gamma)$ contains a closed subspace linearly isomorphic to $L^1(\lambda_1)$. Since any separable subspace of $l^1(\Gamma)$ is in $l^1(\Lambda)$ where $\Lambda \subset \Gamma$ has cardinality $\aleph_0$, $l^1(N)$ contains a subspace linearly isomorphic with $L^1(\lambda_1)$ which is impossible by the remarks on p. 123 of [9]. □

**Proposition 3.** If $E$ is a triple conjugate space with $E''$ linearly isomorphic to an $L$-space then $E$ is linearly isomorphic to an $L$-space.

**Proof.** Let $G''' = E$. $E''$ is an $\ell_1$ space so by [9, II.5.8] $G'$ and $E$ are $\ell_1$ spaces whereas $G$, $G''$, and $E'$ are $\ell_\infty$ spaces. From [9, II.5.7] it follows that $G''$ is injective in Ban. Haydon, [5] shows that a bidual space injective in Ban is linearly isomorphic to $l^\infty(\Gamma)$ for some $\Gamma$. Thus, $E$ is linearly isomorphic to the $L$-space $l^{\infty*}(\Gamma)$. □

**Remarks.** 3. If $F$ is an $\ell_\infty$ space then $F''$ is a bidual space injective in Ban. Thus $F''$ and all higher even order duals of $F$ are linearly isomorphic to $M$-spaces with the odd order duals linearly isomorphic to $L$-spaces. If $E$ is an $\ell_1$ space then $E[2n]$ is linearly isomorphic to an $L$-space and $E[2n-1]$ is linearly isomorphic to an $M$-space if $n > 2$.

2. The main result. To establish our result we will show that the dimension of an infinite dimensional $L$-space is strictly less than that of its bidual. As in [13], the dimension of an infinite dimensional Banach space $F$, $\dim(F)$, is the minimum cardinality of a total subset. Equivalently, $\dim(F)$ is the minimum cardinality of a dense subset, the minimum algebraic dimension of a dense subspace, or the maximum cardinality of a subset, $T$, distinct elements of which are at least a distance 2 apart.

To avoid cumbersome symbolism we shall denote, for cardinal numbers $n$
and $m, n^m$ by either $\exp(n, m)$ or $\exp^i(n, m)$. If $k \in N$ we define, by induction, $\exp^{k+1}(n, m)$ to be $\exp(n, \exp^k(n, m))$.

If $K$ is a compact Hausdorff space and $\mathcal{M}(K) = C'(K)$ then $\dim(\mathcal{M}(K)) > \text{card}(K)$ for the set $\delta(K) = \{\delta_x, x \in K\}$ is a subset such that $\|\delta_x - \delta_y\| = 2$ if $x \neq y$.

If $m$ is a cardinal number we let $2^m = \{0, 1\}^m$ and set $\lambda^m$ equal to the fair coin toss measure on $2^m$. If $\lambda_m$ is the Lebesgue product measure on $[0, 1]^m$ it is known that $L^1(\lambda_m)$ is isometric with $L^1(\lambda^m)$ hence with $L^1(\lambda^m)$ when $m$ is infinite. When $m$ is infinite $\dim(L^1(\lambda_m))$ is known to be $m$ hence $\dim(L^1(\lambda^m)) = m$.

If $\{E_a\}$ is a family of Banach spaces, the $l^1$-direct sum, $(\Sigma_a E_a)_1$, is the set of all $e = (a_\alpha) \in \prod_a E_a$ such that $\|e\|_1 = \Sigma_a \|a_\alpha\| < \infty$. The $l^\infty$-direct sum, $(\Sigma_a E_a)_\infty$, is the set of all $e = (a_\alpha) \in \prod_a E_a$ such that $\|e\|_\infty = \sup_a \|a_\alpha\| < \infty$. $(\Sigma_a E_a)_1$, with the norm $\|\cdot\|_1$, is a Banach space whose dual is $(\Sigma_a E_a)_\infty$ with the norm $\|\cdot\|_\infty$. The dimension of $(\Sigma_a E_a)_1$ is $\Sigma_a \dim(E_a)$ where $\dim(E_a)$ is the algebraic dimension of $E_a$ if $E_a$ is finite dimensional, for, if $T_a$ is total in $E_a$ for all $a$ then $\bigcup_a T_a$ is total in $(\Sigma_a E_a)_1$. If $\Gamma$ is a set with cardinality $m$ then $\dim(l^1(\Gamma)) = m$ for $l^1(\Gamma)$ is an $l^1$-direct sum of one dimensional spaces.

If $G$ is a Banach space with $m = \dim(F) > \aleph_0$ and $T$ is a total subset of $G$ then $G'$ maybe considered to be a subset of $(-\infty, \infty)^T$. Since

$$\text{card}((-\infty, \infty)^T) = \exp(\exp(2, \aleph_0), m) = \exp(2, m)$$

when $m > \aleph_0$ then $\dim(G') < \text{card}(G') < \exp(2, m)$. Similarly, $\dim(G''') < \exp^3(2, m)$ and, in general, $\dim(G^{(k)}) < \exp^k(2, m)$ for all $k \in N$.

If $m > \aleph_0$ is the cardinal of a set $\Gamma$ then $l^\infty(\Gamma)$ is isometric with $C(\beta\Gamma)$ hence

$$\dim(l^\infty(\Gamma)) = \dim(\mathcal{M}(\beta\Gamma)) > \text{card}(\beta\Gamma) = \exp^2(2, m).$$

Since $l^\infty(\Gamma) = [l^1(\Gamma)]^*$, $\dim(l^\infty(\Gamma)) = \exp^2(2, m)$.

If $m > \aleph_0$ then $C(\beta^m)$ is isometric to a subspace of $L^\infty(\lambda^m)$ hence $\mathcal{M}(\beta^m)$ is isometric to a quotient space of $L^\infty(\lambda^m)$. Thus $\dim(L^\infty(\lambda^m)) > \dim(C(\beta^m)) > \text{card}(\beta^m) = 2^m$.

**Proposition 4.** If $E$ is an $L$-space with $m = \dim(E) > \aleph_0$ then $\exp(2, m) < \dim(E''') < \exp^3(2, m)$.

**Proof.** It is only necessary to show that $\exp(2, m) < \dim(E''')$.

We may assume by the Kakutani-Maharam Representation Theorem, [14], that $E = [l^1(\Gamma) + \Sigma_{\alpha \in \Lambda} L^1(\lambda^m)]_1$ with $\Gamma$ and $\Lambda$ sets and $m_\alpha > \aleph_0$ for $\alpha \in \Lambda$.

**Case 1.** $m = \text{card}(\Gamma)$. $E$ contains $l^1(\Gamma)$ isometrically hence $E''$ contains $l^\infty(\Gamma)$ isometrically. Thus

$$\dim(E''') > \dim(l^\infty(\Gamma)) = \exp^2(2, m) > \exp(2, m).$$

**Case 2.** $m = m_\alpha$ for some $\alpha \in \Lambda$. $E$ contains $L^1(\lambda^m)$ isometrically hence

$$\dim(E''') > \dim(L^\infty(\lambda^m)) > \exp(2, m).$$
Case 3. \( m > \text{card}(\Gamma) \) and \( m > m_\alpha \) if \( \alpha \in \Lambda \). In this case

\[
m = \sum_{\alpha \in \Lambda} m_\alpha = \sup_{\alpha \in \Lambda} m_\alpha.
\]

Regard each \( m_\alpha \) as the first ordinal of cardinal \( m_\alpha \). Well order \( \Lambda \) so that \( \{ m_\alpha : \alpha \in \Lambda \} \) is nondecreasing along \( \Lambda \). If the ordinals \( m \) and \( m_\alpha \) are regarded as sets then \( m \) is the increasing union \( \bigcup_{\alpha \in \Lambda} m_\alpha \). Let \( \pi_\alpha : \mathcal{2}^m \to \mathcal{2}^{m_\alpha} \) be the natural projection restricting a function on the set \( m \) to the smaller set \( m_\alpha \). Let \( \mathcal{G}_\alpha \) be the Borel sets in \( \mathcal{2}^{m_\alpha} \), let \( \mathcal{G} \) be the Borel sets in \( \mathcal{2}^m \) and let \( \mathcal{G}_\alpha = \pi_\alpha^{-1}(\mathcal{G}_\alpha) \) for \( \alpha \in \Lambda \). The set \( \{ \mathcal{G}_\alpha : \alpha \in \Lambda \} \) is an increasing family of \( \sigma \)-algebras. The inverse image of the measure \( \lambda^{m_\alpha} \) on \( \mathcal{G}_\alpha \) under \( \pi_\alpha \) is the restriction of \( \lambda^m \) to \( \mathcal{G}_\alpha \). Equivalently, \( \lambda^{m_\alpha} \) is the image of \( \lambda^m \) on \( \mathcal{2}^{m_\alpha} \) under \( \pi_\alpha \). The map \( j_\alpha : f \to f \circ \pi_\alpha \) yields an isometric embedding of \( L^1(\lambda^{m_\alpha}) \) into \( L^1(\lambda^m) \) with range \( L^1(\mathcal{2}^{m_\alpha}, \mathcal{G}_\alpha, \lambda^{m_\alpha}) \). If \( f = (f_\alpha) \in [\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1 \) define \( j(f) = \Sigma_{\alpha \in \Lambda} j_\alpha(f_\alpha) \). The map \( j \) is of norm 1 from \( [\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1 \) into \( L^1(\lambda^m) \). If \( g \in L^1(\lambda^m) \) and \( \alpha \in \Lambda \) then \( E(g|\mathcal{G}_\alpha) \) is in the range of \( j_\alpha \) hence is in that of \( j \) and \( \|E(g|\mathcal{G}_\alpha)\| < \|g\| \). By the Martingale Convergence Theorem,

\[
\lim_{\alpha \in \Lambda} E(g|\mathcal{G}_\alpha) = g
\]

in \( L^1(\lambda^m) \). This shows that the image of the unit ball in \( [\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1 \) is dense in that of \( L^1(\lambda^m) \). From this it follows that \( j \) is a surjection and that \( L^1(\lambda^m) \) is isometric to the quotient of \( \Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha}) \) by \( \ker(j) \) hence that \( L^1(\lambda^m) \) is isometric to a quotient space of \( E^\infty \). Thus, \( L^1 \) is isometric with a quotient space of \( E^\infty \) and \( \dim(E^\infty) > \dim(L^1(\lambda^m)) \geq \exp(2, m) \).

\textbf{Corollary 4-1.} Let \( E \) be an \( L \)-space with \( m = \dim(E) > \aleph_0 \). If \( k \in \mathbb{N} \) then \( \exp^k(2, m) \leq \dim(E^{[2^k]}) \leq \exp^{2k}(2, m) \).

\textbf{Corollary 4-2.} If \( E \) is an \( L \)-space with \( \aleph_0 < \dim(E) \leq \exp^k(2, \aleph_0) \) for some \( k \in \mathbb{N} \) then \( n(E) < 2k \).

\textbf{Corollary 4-3.} If \( E \) is an \( L \)-space with \( \dim(E) \geq \exp^k(2, \aleph_0) \) for all \( k \in \mathbb{N} \) then \( E \) has no separable predual of any order.

\textbf{Corollary 4-4.} If \( F \) is an infinite dimensional Banach space and \( k \in \mathbb{N} \) with \( F^{[k]} \) linearly isomorphic to an \( L \)-space then \( F^{[n]} \) and \( F^{[m]} \) are not linearly isomorphic for any \( n \neq m \) in \( \mathbb{N} \).

This last corollary is very well known.

\textbf{Corollary 4-5.} If \( E \) is an infinite dimensional \( L \)-space it is nonreflexive.

\textbf{Proposition 5.} If \( E \) is an infinite dimensional \( L_1 \) (or \( L_\infty \) space) there is an integer \( \bar{n}(E) < \infty \) so that if \( F \) is linearly isomorphic to \( E \) then \( n(F) < \bar{n}(E) \).

\textbf{Proof.} Let \( C \) be the set of cardinal numbers \( m \) so that there is an \( L \)-space \( F \) with \( E \) linearly isomorphic to \( F^{[2n]} \) for some integer \( n \) and with \( m = \dim(F) \). Let \( M = \min(C) \). Let \( k_0 \) be an integer and let \( F_0 \) be an \( L \)-space with \( \dim(F_0) = M \) and with \( F_0^{[2k_0]} \) linearly isomorphic to \( E \). If \( G \) is any \( L \)-space
and $j$ is an integer with $G^{[2j]}$ linearly isomorphic with $E$, Corollary 4-1 and the fact that $\dim(G) \geq M$ implies that
\[
\exp^j(2, M) < \dim(G^{[2j]}) = \dim(E) = \dim(F_0^{[2k_0]}) < \exp^{2k_0}(2, M).
\]
Thus $j < 2k_0$.

If $E$ is not isomorphic with a fourth conjugate space then $\bar{n}(E) < 3$. Otherwise, if $E$ is linearly isomorphic to $F^{[2k]}$ with $k > 2$ Proposition 3 and Remarks 3 show that $F^{[4]}$ is isomorphic to an $L$-space $G$. Thus $k - 2 < 2k_0$. Allowing for the possibility that $F$ is a dual space it follows that $\bar{n}(E) < 2k_0 + 3$. □

**Corollary 5-1.** If $E$ is an $L$-space (or $M$-space) then $n(E) < \infty$.

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**References**


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